Noise-Induced Transitions in a Simplified Model of the Thermohaline Circulation

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ABSTRACT

A simplified box ocean model for the North Atlantic is used to study the influence of multiplicative short-term climate variability on the stability and long-term dynamics of the North Atlantic thermohaline circulation. A timescale separation between fast temperature and slow salinity fluctuations is used to decouple the dynamical equations resulting in a multiplicative stochastic differential equation for salinity. As a result the qualitative behavior and the stability of the thermohaline circulation become a function of the noise level. This can be understood in terms of the concept of noise-induced transitions. Furthermore, the role of nonvanishing noise autocorrelation times on the dynamics of the thermohaline circulation is investigated. Red noise temperature forcing generates new equilibria, which do not have a deterministic counterpart. This study suggests that noise-induced transitions might have climate relevance.

1. Introduction

Climate reconstructions from glacier and deep ocean sediment cores indicate that the climate system is capable to undergo very rapid transitions. In particular the climate reconstructions during the Last Glacial period from about 80 ka BP to 10 ka BP (ka ≡ kiloyears) show rapid[1] transitions from a cold to a warm climate state and back. These so-called interstadial transitions also known as Dansgaard–Oeschger events occur on millenial timescales (Dansgaard et al. 1993; Bond et al. 1997; Grootes and Stuiver 1997). Whether these transitions are triggered by an external forcing or whether they are generated by internal climate instabilities is still unknown. However, the thermohaline circulation (THC) is expected to play an important role in this context. Evidence is reported both from observations (Boyle and Keigwin 1982, 1987; Crowley 1983; Sarntheim et al. 1996) and from climate models of different complexity (Stommel 1961; Manabe and Stouffer 1988; Rahmstorf 1995) that the THC exhibits several modes of operation,

which could possibly alternate on millenial timescales. One prominent candidate for such a transition from one mode of operation to another is the Younger Dryas event (11 ka BP). It is argued (Broecker et al. 1985) that this event was triggered by a massive discharge of meltwater into the North Atlantic thereby reducing the meridional overturning circulation. Eventually this led to a reduction of the poleward heat transport and a dramatic change in particular of the European climate conditions. It is well established that the climate mean state determines the variability. The reverse direction as illustrated schematically in Fig. 1 is not so well explored and is the focus of this investigation. In order to understand the past climate history and also the sensitivity of the climate system to an anticipated anthropogenic increase of greenhouse gas concentrations in the atmosphere it is necessary to identify and understand those factors that control the thermohaline circulation. In this context the question of how different climate model parameters influence the simulated climate statistics is a very crucial one. It is argued that the sensitivity of the overturning circulation simulated in GCMs depends on factors such as the formulation of the boundary conditions (Maier-Reimer and Mikolajewicz 1993; Mikolajewicz and Maier-Reimer 1994; Bryan and Hansen 1995; Lohmann et al. 1996) and the representation of specific physical processes (Marotzke and Stone 1995; Nakamura et al. 1994; Lohmann 1998). Furthermore, it was discussed

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[1] "Rapid" here denotes a timescale range of a few years up to decades.

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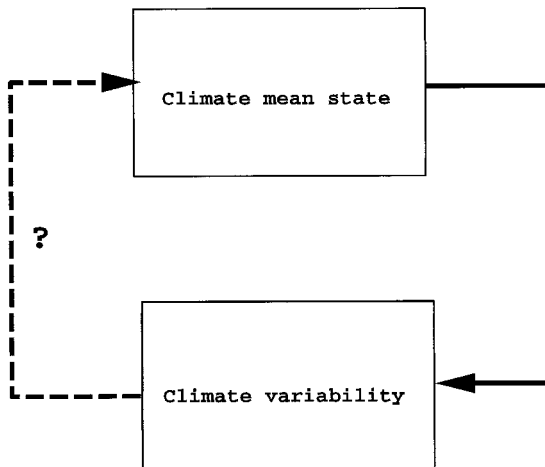E-mail: timmera@knmi.nl

FIG. 1. Schematic view of statistical interactions in the climate system.

by Cessi (1994) that also the variance of some stochastic buoyancy forcing component modifies the stability properties of the THC. A threshhold fresh-water noise level was identified in this study that regulates the residence in one of the potential wells of the THC. One of the main objectives of our paper is to illustrate that noise does not only regulate the transition from one equilibrium to another but is also responsible for the occurence of new equilibria.

Apart from a detailed modeling of the climate system using coupled general circulation models, sometimes very simplified climate models could give insight into the general characteristics of our climate system. The most simplified climate models can be described in terms of stochastic differential equations (Hasselmann 1976; Oerlemans 1979; Benzi et al. 1982; Nicolis and Nicolis 1981; Nicolis 1982; Ruiz de Elvira and Lemke 1982; Blaauboer et al. 1982; Müller 1987; Bryan and Hansen 1993; Cessi 1994; Lohmann and Schneider 1999). These stochastic climate models can be viewed as comprehensive paradigms or metaphors for particular features of the climate system. Some of these models focus mainly on the properties of linear stationary and cyclostationary Langevin equations (Frankignoul and Hasselmann 1977; Ruiz de Elvira and Lemke 1982; Oerlemans 1979), whereas other stochastic models also account for the inherently nonlinear nature of the climate system. Such nonlinearities could arise for example from the nonlinearity of the atmospheric energy balance (Benzi et al. 1982; Matteucci 1989), the multistability of the THC (Stommel and Young 1993; Bryan and Hansen 1993; Cessi 1994), or, in a certain sense, also by multiplicative stochastic forcing[2] (Blaauboer et al. 1982;

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[2] A linear univariate multiplicative stochastic differential equation can always be transformed into a nonlinear additive stochastic differential equation.

Müller 1987). Our approach here is a dynamical system approach standing in this tradition. We intend to investigate the characteristics of the thermohaline circulation with respect to different stochastic forcing scenarios in a nonlinear mathematical framework.

In section 2 we give a brief description of the simplified stochastic climate model used in this study and its dynamics. It is shown that a timescale separation *ansatz* decouples the dynamical equations of the box model. The remaining univariate stochastic salinity equation is analyzed in terms of stochastic stability. Section 3 is devoted to the analysis of the Langevin equation describing the salinity dynamics in a red noise environment. The effect of nonvanishing autocorrelation times on the gross climate statistics is discussed in a unified colored-noise framework. In section 4 we give a summary and discussion of our results.

2. Model description

The simplified model used here is a box model that mimics the basic features of the North Atlantic meridional ocean overturning, which is partly responsible for the oceanic meridional heat transport.

Box models of various kinds have been used in different studies by Stommel and Young (1993), Bryan and Hansen (1993), Cessi (1994), Lohmann et al. (1996), Prange et al. (1997), Lohmann and Gerdes (1998), and Lohmann and Schneider (1999).

The box ocean model used here is based on the original ideas of Stommel (1961). It consists of a low-latitude ocean box characterized by the temperature T_1 and the salinity S_1, and a high-latitude ocean box characterized by the temperature T_2 and the salinity S_2. The depth of these boxes is denoted h. The meridional differences ΔS and ΔT are defined as $S_1 - S_2$ and $T_1 - T_2$. Furthermore, we introduce the quantity Φ, which represents the horizontal mass exchange due to the density difference between the two boxes. In our study we adopt the Stommel (1961) ansatz: Φ is given by $\Phi = c|\alpha\Delta T - \beta\Delta S|$, where the thermal and haline expansion coefficients of water are expressed as $\alpha = 0.15$ K^{-1} and $\beta = 0.8$ psu^{-1}. Here c is a tunable parameter. The dynamical equations for salinity can be written as

$$\partial_t S_1 = -\frac{\Phi}{2V}\Delta S + \frac{S_0}{2h}(P - E) \qquad (1)$$

$$\partial_t S_2 = +\frac{\Phi}{2V}\Delta S - \frac{S_0}{2h}(P - E). \qquad (2)$$

Similarly the prognostic equations for the box temperatures read

$$\partial_t T_1 = -\frac{\Phi}{2V}\Delta T + \frac{F_{\text{oa}}}{2c_p\rho h} \qquad (3)$$

$$\partial_t T_2 = +\frac{\Phi}{2V}\Delta T - \frac{F_{\text{oa}}}{2c_p\rho h}, \qquad (4)$$

where V denotes the volume of the ocean box, $(P - E)$ the difference between precipitation and evaporation, and F_{oa} the heat flux at the ocean–atmosphere interface. Furthermore, we introduced the reference salinity $S_0 = 35$ psu and the heat capacity $\rho_0 c_p h$. Subtracting the prognostic equations for the individual boxes yields

$$\partial_t \Delta S = -\frac{c}{V}|\alpha\Delta T - \beta\Delta S|\Delta S + \frac{S_0}{h}(P - E) \quad (5)$$

$$\partial_t \Delta T = -\frac{c}{V}|\alpha\Delta T - \beta\Delta S|\Delta T + \frac{F_{\text{oa}}}{c_p\rho h}. \qquad (6)$$

One of the essential findings of Lohmann and Schneider (1999) is that changes of the thermohaline circulation are governed by temperature and salinity fluctuations that act on different timescales. The thermally driven regime is associated with fast temperature fluctuations, whereas the timescale of the salinity variations is much longer. This is due to the fact that the atmosphere "feels" SST anomalies. After the generation of a SST anomaly in midlatitudes the heat flux (Frankignoul 1985) tends to damp it. An analogy for salinity does not exist because the atmosphere is insensitive to the presence of salinity anomalies. We end up with a typical timescale separation problem (Hasselmann 1977). On short timescales the temperature dynamics is mainly governed by the stochastic heat flux component. Changes of the meridional overturning rate are assumed to be small $\Phi \sim \Phi_0 = c|\alpha\Delta T_0 - \beta\Delta S_0|$. The nulled quantities represent typical mean values. Under this assumption and using the stochastic heat flux ansatz $F_{\text{oa}} \sim \sigma\xi_t$ (6) reduces to the Langevin equation for an Ornstein–Uhlenbeck process:

$$\partial_t \Delta T = -\gamma\Delta T + \sigma\xi_t, \qquad (7)$$

where ξ_t represents Gaussian white noise of variance 1 and $\gamma = c|\alpha\Delta T_0 - \beta\Delta S_0|/V$; σ measures the standard deviation of the external stochastic component. Temperature anomalies due to ocean dynamics are neglected and only the mixed layer thermodynamics remains.[3] Mixed layer temperature anomalies as simulated by Eq. (7) can be described approximately by a red noise spectrum (Frankignoul and Hasselmann 1977) proportional to $(\Omega^2 + \gamma^2)^{-1}$; Ω denotes the frequency. For large γ

the spectrum can be approximated by a white noise spectrum. Hence, in our first assumption temperature anomalies will regarded as Gaussian white noise $\Delta T = \Delta T_0 + \sigma\xi_t$, whereas in a second step (section 3) we investigate the more realistic red noise case. Rescaling (5) and (7) by $\Delta S = \alpha\Delta T_0 y/\beta$ and $t = V/(c\alpha\Delta T_0)\hat{t}$, substituting the white noise assumption into (5), and using the stochastic freshwater assumption of Stommel and Young (1993)

$$\frac{S_0}{h}(P - E)\frac{\beta V}{c\alpha^2(\Delta T_0)^2} = \mu_0 + \theta\zeta_t \qquad (8)$$

we obtain the new (Langevin) equation

$$\partial_{\hat{t}} y = f(y) + g(y)\xi_{\hat{t}} \qquad (9)$$

with

$$f(y) = -|1 - y|y + \mu_0 \quad \text{and}$$

$$g(y) = \Sigma(y + \omega). \qquad (10)$$

Here $\Sigma \equiv \sigma\sqrt{c\alpha/(\Delta T_0 V)}$, $\langle\xi_{\hat{t}}, \xi_{\hat{t}'}\rangle \equiv \delta(\hat{t} - \hat{t}')$ and $\omega \equiv \theta/\Sigma$ were used, where θ characterizes the magnitude of the additive stochastic freshwater component whereas μ_0 represents the mean freshwater flux difference between high and low latitudes. The assumption was made that freshwater noise $\zeta_{\hat{t}}$ and temperature noise $\xi_{\hat{t}}$ are correlated.[4]

Comparing our Eq. (9) with Eq. (1.4) in Stommel and Young (1993) and Eq. (2.8b) in Cessi (1994) we see that the crucial difference to these studies is the multiplicative noise component $g(y)\xi_{\hat{t}}$.

The multiplicative Langevin equation (9) describes a diffusion process. An alternative (or equivalent) way of describing the dynamics of diffusion processes is the Fokker–Planck equation [Gardiner 1997; Stommel and Young (1993) Eq. (3.3) and Cessi (1994) Eq. (4.4)]:

$$\partial_t p(y) = -\partial_y\left\{\left[f(y) + \frac{\Sigma^2}{2}g(y)\partial_y g(y)\right]p\right\}$$

$$+ \frac{\Sigma^2}{2}\partial_{yy}[g^2(y)p], \qquad (11)$$

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[3] Expressing our methodology in a more theoretical manner: the fast temperature dynamics allow for the application of the center manifold theory and elimination of the fast variable. This technique (slaving principle) provides quite a general mechanism for multiplicative noise.

[4] The thermohaline circulation is driven partly by convective processes in the Nordic seas. These processes are very much dependent on the local buoyancy fluxes. The oceanic processes in the Labrador Sea, as one of the main sinking regions in the North Atlantic, are governed by cold arctic air outbreaks. This implies that less precipitation goes along with cold temperatures and vice versa. Furthermore, under the same atmospheric conditions warm ocean temperatures in the Nordic seas are associated with strong evaporation, thereby changing also the freshwater budget. This has been tested using a multicentury coupled GCM integration performed with ECHAM3/LSG (Voss et al. 1998; Timmermann et al. 1998). The correlation between the area-averaged freshwater anomalies and the upper ocean temperatures in the Labrador Sea based on 3-yr mean values is 0.6. Hence, on a climatological timescale the assumption of coherent freshwater and temperature fluctuations seems a reasonable approximation.
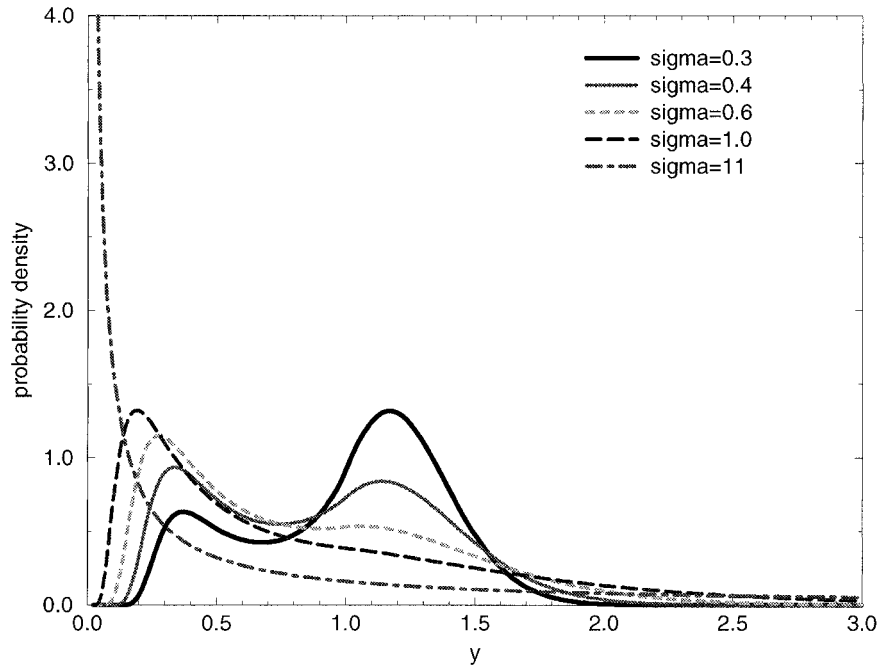
FIG. 2. Probability distribution of salinity anomalies as obtained from Eqs. (13) and (14) for μ_0 = 0.25; Σ = 0.3, 0.4, 0.6, 1, 11.

with $p(y)$ designating the probability density to obtain the value y. Here we are interested only in the stationary probability density. It can be obtained from (11) by setting the left side to zero and integrating twice with respect to y. The result is (see Horsthemke and Lefever 1984)

$$p_s(y) = Ng(y)^{-1} \exp\left[\frac{2}{\Sigma^2} \int^y f(u)/g^2(u) \, du\right], \quad (12)$$

which in case of (9) and for $\omega = 0$ and $y > 0$ reads explicitly:

$$p_s^A(y) = N_1 y^{-2\Sigma^{-2}-1} \exp\left[2\Sigma^{-2}\left(y - \frac{\mu_0}{y}\right)\right]$$

$$\text{for } 0 < y < 1 \quad (13)$$

$$p_s^B(y) = N_2 y^{2\Sigma^{-2}-1} \exp\left[2\Sigma^{-2}\left(-y - \frac{\mu_0}{y}\right)\right]$$

$$\text{for } 1 < y. \quad (14)$$

This last assumption holds in a present day climate. We choose $\mu_0 = 0.25$, that is, the critical value above which the deterministic bistability vanishes (Stommel and Young 1993). Here N_2 can be obtained from the continuity condition $p_s^A(1) = p_s^B(1)$. Figure 2 displays $p_s(y)$ for different noise levels Σ = 0.3, 0.4, 0.6, 1.0, 11.0. One observes that the qualitative structure changes as a function of the multiplicative noise level. Apart from a change of the most probable value a change in the

overall shape of $p_s(y)$ becomes apparent. Noise destroys the bimodality of the probability density. The most probable value shifts from about 1.2 to values close to zero. Hence, temperature noise has a destabilizing effect on the present day THC in the simplified model discussed here. Strong multiplicative noise squeezes the system into a state characterized by small y values. The characteristic qualitative change of the probability density is called "noise-induced transition" (NIT: Horsthemke and Lefever 1984). The shift of the most likely value of y can easily be obtained from $\partial_y p_s(\overline{y}) = 0$, which yields $f(\overline{y}) - \Sigma^2/2g(\overline{y})g'(\overline{y}) = 0$. The result for $\overline{y}$ is shown in Fig. 3 as a function of noise level. One observes a typical saddle node bifurcation of the system. Small changes in the noise level close to the critical noise level $\Sigma_{\text{crit}} = 2^{-1/2}$ (indicated by an arrow) can induce a catastrophic regime transition. For relatively large μ_0 the distribution is unimodal, and the resulting shift in the most likely value is similar to the dark solid curve in Fig. 3. The bifurcation diagram is reduced to one stable solution for all Σ. For relatively small μ_0 the bifurcation diagram exhibits two stable equilibria and one unstable equilibrium similar to the subcritical behavior as depicted in Fig. 3. Combining this information, the characteristic behavior of our system is reminiscent of a cusp catastrophe graph in the (μ_0, Σ) plane (Thom 1975) as illustrated in Fig. 4. Note the characteristic hysteresis curve, allowing for smooth and abrupt transitions.

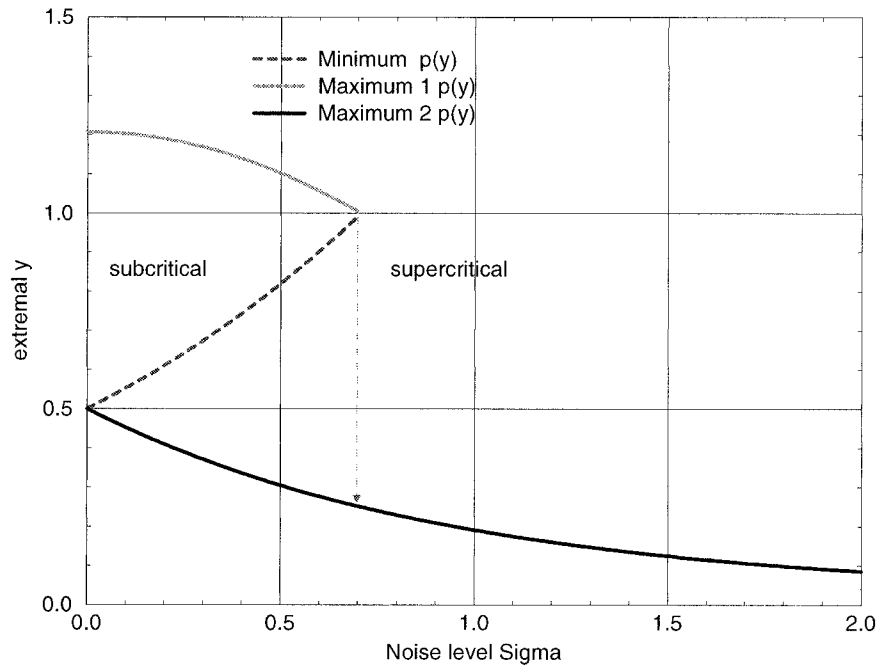The multiplicative stochastic component affects also

FIG. 3. Relative maxima and minima of Eqs. (13) and (14) as a function of the noise level Σ; $\mu_0 = 0.25$. The qualitative behavior of the Stommel model as a function of noise amplitude is characterized by a saddle node bifurcation.

the system's sensitivity to perturbations. The sensitivity of the system can be characterized by its Lyapunov exponents (Oseledets 1968; Carverhill 1985):

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \ln|v(t)|, \qquad (15)$$

where $v(t)$ denotes the tangent linear term as obtained from

$$\partial_t v = \partial_y [f(y) + \Sigma g(y)\xi_t]v. \qquad (16)$$

This Langevin equation has to be transformed into a stochastic differential equation, which (in Ito calcul) reads

$$\frac{dv}{v} = \left[f'(y) + \Sigma g'(y)\xi_t + \frac{\Sigma^2}{2} g'(y)g''(y) \right] dt. \qquad (17)$$

In this case ξ_t has to be regarded as an Ito process. A more detailed explanation of these subtleties, however, exceeds the scope of this paper. The linear dynamics of perturbations $v(t)$ is explicitly given by

$$v(t) = v(0) \exp\left\{ \int_0^t \left[f'(y) + \Sigma g'(y)\xi_{t'} + \frac{\Sigma^2}{2} g'(y)g''(y) \right] dt' \right\}. \qquad (18)$$

Due to ergodicity, the time average can be substituted by the ensemble average $\mathbf{E}$, and the Lyapunov exponent is

$$\lambda = \mathbf{E}(\ln|v(t)|)$$

$$= \int \left[f'(y) + \frac{\Sigma^2}{2} g'(y)g''(y) \right] p_s(y)\, dy$$

which yields in our case

$$\lambda = \int_0^1 (-1 + 2y)p_s^A(y)\, dy + \int_0^\infty (1 - 2y)p_s^B(y)\, dy. \qquad (19)$$

This integral is calculated numerically using equations (13), (14) and adopting the Patterson (1968) technique. In Figure 5 we observe negative Lyapunov exponents for the whole integrable range of Σ. Initial perturbations will be damped out according to an exponential law $\sim e^{\lambda t}$. Figure 5 depicts also a relative minimum of λ for $\Sigma \sim 0.4$–0.5. The noise level $\overline{\Sigma}$ which maximizes the Lyapunov exponent can also be computed analytically from $\partial_\Sigma \lambda(\overline{\Sigma}) = 0$.

Our analytical analysis has illustrated that the simple box model, which mimics the North Atlantic overturning circulation is able to undergo NITs. Furthermore,

this analysis has revealed that *in general* Lyapunov stability depends crucially on the multiplicative noise component due to the term $\frac{1}{2}\Sigma^2 g'(y)g''(y)$ (see above).

3. The effect of colored multiplicative noise

Here we present a first-order extension to the problem discussed above, in particular a more general statistical approach which is not based on the white noise approximation for temperature fluctuations. The fact is taken into account that upper-ocean temperature variability on climatological timescales can to first order be characterized by an Ornstein–Uhlenbeck process with a red noise spectrum (Hasselmann 1976). Hence, we make the following ansatz for the random environmental fluctuations:

$$\langle \epsilon_t \epsilon_s \rangle = \frac{\Sigma^2}{\tau} e^{-|t-s|/\tau},$$

where angle brackets designate the ensemble mean value. A problem arises in the statistical description of processes driven by colored noise: The stochastic dynamics can no longer be represented as a Markov process and the powerful methods of Markov processes can no longer be applied. An approximate solution of the problem is provided by the unified colored noise approximation (UCNA) (Jung and Hänggi 1987) presented here.

In the following we investigate the salinity equation (9) in a multiplicative red noise environment. For simplification we again drop the additive freshwater noise component ω and obtain

$$\partial_t y = -|1 - y|y + \mu_0 + y\epsilon_t \tag{20}$$

$$\langle \epsilon_t \epsilon_s \rangle = \frac{\Sigma^2}{\tau} e^{-|t-s|/\tau}. \tag{21}$$

The multiplicative Langevin equation (20) can be transformed into an additive system by the following transformation $z = \ln y$. One ends up with the two-dimensional Markovian flow equation

$$\partial_t z = -|1 - e^z| + \mu_0 e^{-z} + \epsilon_t = f(z) + \epsilon_t \tag{22}$$

$$\partial_t \epsilon = \frac{-\epsilon}{\tau} + \frac{\Sigma}{\tau} \xi_t \tag{23}$$

$$\langle \xi_t \xi_s \rangle = \delta(t - s). \tag{24}$$

Adopting the unified colored-noise approach (Jung and Hänggi 1987) we calculate the derivative of Eq. (22) with respect to the scaled time $t = \sqrt{\tau}\tilde{t}$. This yields

$$\partial_{\tilde{t}\tilde{t}} z + \kappa(z, \tau)\partial_{\tilde{t}} z - f(z) = \Sigma \xi(\sqrt{\tau}\tilde{t}) \tag{25}$$

$$\kappa(z, \tau) = \frac{1}{\sqrt{\tau}} - \sqrt{\tau}\partial_z f(z). \tag{26}$$

If the second term in (25) is large as compared to the first term, one can use the adiabatic approximation ($\partial_{\tilde{t}\tilde{t}} z$

= 0) to remove fast fluctuations of the system (Haake 1982; Gardiner 1997). Eliminating further the red noise fluctuation by Eq. (24), one obtains the multiplicative Langevin equation

$$\partial_{\tilde{t}} z = \frac{f(z)}{\kappa(z, \tau)} + \left[\frac{\Sigma \tau^{-1/4}}{\kappa(z, \tau)} \right] Q(\tilde{t}), \tag{27}$$

driven by white noise with $\langle Q(\tilde{s})Q(\tilde{t}) \rangle = \delta(\tilde{s} - \tilde{t})$. Integrating the corresponding Fokker–Planck equation (in the Stratonovich calcul) gives the following stationary adiabatic probability density

$$p_s(z, \tau) = N|1 - \tau\partial_z f(z)|$$

$$\times \exp\left[\frac{2}{\Sigma^2}\left(-\frac{1}{2}\tau f^2(z) + \int^z f(u)\,du \right) \right]. \tag{28}$$

This is the fundamental finding of the UCNA approach of Jung and Hänggi. A back-transformation to the original quantities is provided by $p_s(y, \tau) = p_s(z, \tau)|dz/dy|$. This approximation is exact for $\tau \to 0$ and $\tau \to \infty$. The adiabatic approximation holds only if $\kappa \gg \tilde{t}^{-1}$, where $\tilde{t}^{-1}$ is the normalized timescale of consideration. This condition reads explicitly

$$\kappa(y, \tau)$$

$$= \begin{cases} \left| \dfrac{1}{\sqrt{\tau}} - \sqrt{\tau}\left(y - \dfrac{\mu_0}{y}\right) \right| \gg \tilde{t}^{-1} & \text{for } 0 < y < 1 \quad (29) \\[4mm] \left| \dfrac{1}{\sqrt{\tau}} - \sqrt{\tau}\left(-y - \dfrac{\mu_0}{y}\right) \right| \gg \tilde{t}^{-1} & \text{for } y > 1. \quad (30) \end{cases}$$

Using the renormalization $\hat{t} = \sqrt{\tau}\tilde{t}$ and that the autocorrelation time τ of an Ornstein–Uhlenbeck process is inverse to the damping factor γ [see Eq. (7)] and, hence, $\tau = \gamma^{-1}c\alpha\Delta T_0/V = 1/(1 - y_0)$), we obtain

$$1 - y_0 - y + \frac{\mu_0}{y} \gg \hat{t}^{-1} \tag{31}$$

$$1 - y_0 + y + \frac{\mu_0}{y} \gg \hat{t}^{-1}. \tag{32}$$

Typical values are $y = y_0 = 0.5$ and $\mu_0 = 0.25$, which yields $\hat{t} \gg 2$. Using realistic values $c = 17 \times 10^6$ m³ s⁻¹, $V = 2 \times 10^{15}$ m³ (corresponding to an effective ocean depth of 300 m),[5] $\Delta T_0 = 15$ K the dimensionless time can be retransformed into a dimensional value according to $t = V/(c\alpha\Delta T_0)\hat{t}$. This yields $t \gg 3$ years.

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[5] In nature it is often sufficient to change the density structure in the upper high latitude ocean column in order to change the strength of the overturning circulation. This is due to convection, which is neglected in the Stommel model. In order to account for this effect heuristically we choose an effective depth of 300 m rather than the full ocean depth. Moreover, our approach can be regarded as a combination of stochastic mixed layer dynamics (Hasselmann 1976) and salinity dynamics according to the Stommel (1961) model.
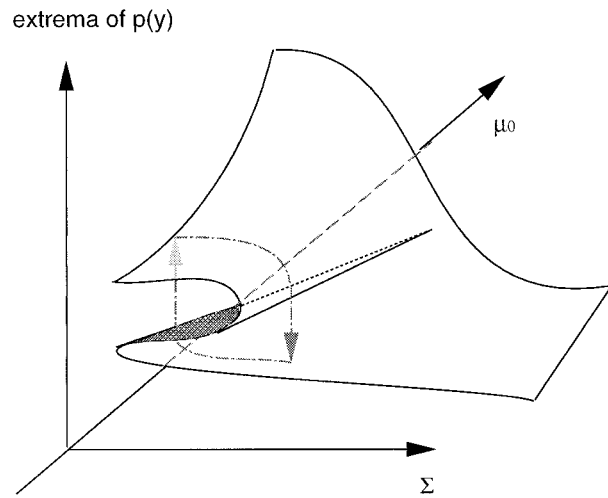
extrema of p(y)

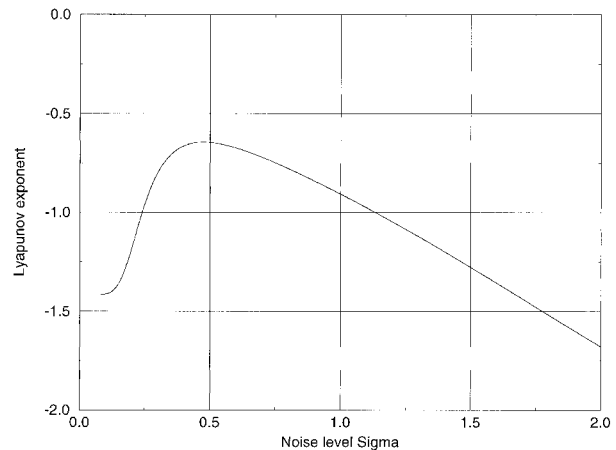FIG. 4. Schematic catastrophe graph for the Stommel model with multiplicative noise.

FIG. 5. Lyapunov exponent as a function of the noise level Σ; μ_0 = 0.25.

Since advective processes in the THC have decadal to centennial timescales, the unified colored noise approach is a reasonable approach to study the longterm dynamics of the THC. Furthermore, typical dimensional values of τ obtained from

$$\tau = V/[(1 - y_0)c\alpha\Delta T_0]$$

are also in the order of 3 years (for $y_0 = 0.5$), which corresponds very well to observed values in the extratropical oceans (Frankignoul and Hasselmann 1977).

The probability density (28) for A: $0 < y < 1$ and B: $1 < y$ reads in our case explicitly

$$p_s^A(y, \tau) = N_1 y^{-1} \left| 1 + \tau\left(\frac{\mu_0}{y} - y\right) \right| \exp\left\{ \frac{2}{\Sigma^2}\left[-\frac{1}{2}\tau\left(-1 + y + \frac{\mu_0}{y}\right)^2 - \ln y + y - \frac{\mu_0}{y} \right] \right\} \tag{33}$$

$$p_s^B(y, \tau) = N_2 y^{-1} \left| 1 + \tau\left(\frac{\mu_0}{y} + y\right) \right| \exp\left\{ \frac{2}{\Sigma^2}\left[-\frac{1}{2}\tau\left(1 - y + \frac{\mu_0}{y}\right)^2 + \ln y - y - \frac{\mu_0}{y} \right] \right\} \tag{34}$$

with the continuity condition

$$N_2 = N_1 \left| \frac{1 - \tau + \mu_0\tau}{1 + \tau + \mu_0\tau} \right| e^{4/\Sigma^2}. \tag{35}$$

The probability density is calculated according to (33) and (34). The result for $\Sigma = 0.3$ is depicted in Fig. 6 for different autocorrelation times $\tau = 0.1, 0.8, 3, 5$, corresponding to dimensional values of 2 months, 16 months, and 5 and 8 years. The probability density for $\tau = 0.1$ matches the white noise curve shown in Fig. 2. Increasing the decorrelation time results in a dramatic change of the shape of the probability density. We observe that a new maximum close to $y = 0.7$ arises as well as two minima close to $y = 0.6$ and $y = 1$. Thus, red noise generates a new stable equilibrium and two unstable equilibria, which do neither, have a deterministic nor a white noise counterpart. It turns out that in our simplified model not only the total variance but also the spectral variance distribution of the stochastic com-

ponent are crucial for the dynamics of the thermohaline circulation. Hence, multiple equilibria are not just a property of the THC but depend also on the stochastic forcing. These results are qualitatively very similar also for supercritical noise levels as illustrated in Fig. 7 for $\Sigma = 1$ and $\mu_0 = 0.25$. In this case nonvanishing decorrelation times can even revitalize an equilibrium that has disappeared in the white noise case due to the high noise variance. Given a long autocorrelation time the thermohaline circulation can jump abruptly from the left stable equilibrium to the right stable equilibrium.

4. Discussion and summary

Using a simplified box ocean model we documented that the stability of the thermohaline circulation is a function of the meridional temperature noise level. This stochastic temperature component could either be interpreted as the representation of unresolved physical
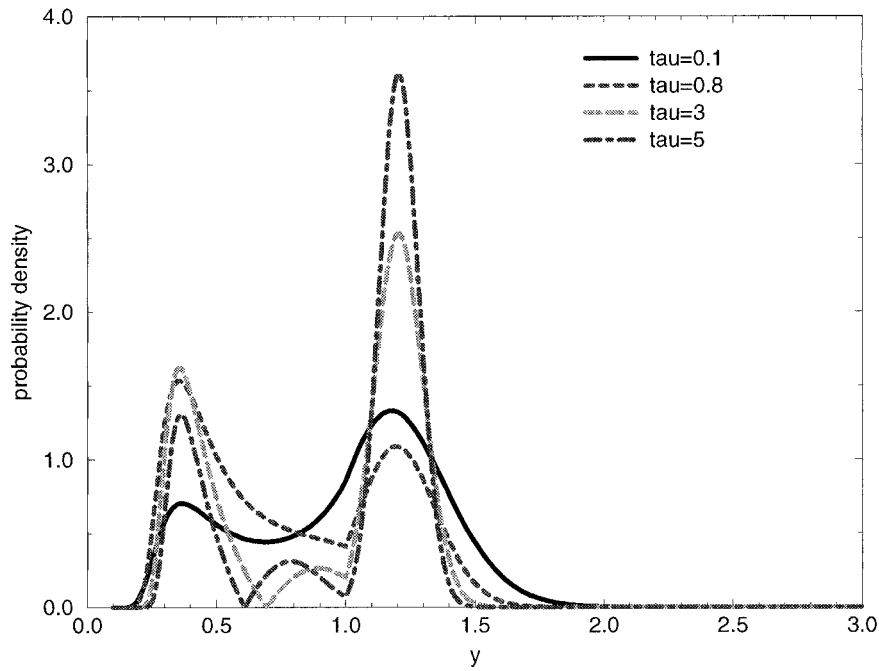
FIG. 6. Probability density of salinity differences as obtained from Eqs. (33) and (34) for $\Sigma = 0.3$, $\mu_0 = 0.25$.

processes or as the expression of a thermohaline time-scale separation. A critical noise level exists that induces an abrupt transition of the salinity statistics (NIT). Low (subcritical) noise levels are associated with two stationary states corresponding to mean haline density gradients of about 1.2 and 0.5. Beyond a critical noise level, which depends on the freshwater forcing offset μ_0, the most probable value shifts immediately to values below 0.5. The concept of noise-induced transitions explains this qualitative behavior. Furthermore, it can be argued
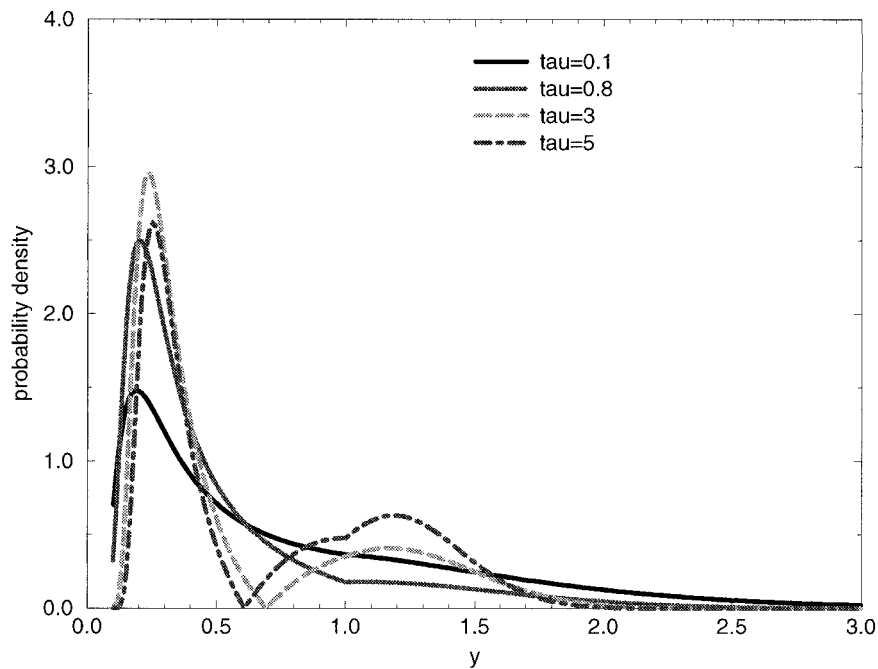

FIG. 7. Probability density of salinity differences as obtained from Eqs. (33) and (34) for $\Sigma = 1.0$, $\mu_0 = 0.25$.

that a changing noise amplitude could lead to rapid changes in the mean state and, in the context of climate variability, to abrupt climate transitions. The concept of noise-induced transitions replaces the question mark in Fig. 1. Furthermore this paper addressed the question of the influence of colored noise on the simulated climate statistics. Using the UCNA approach it has been found that the autocorrelation time of the driving noise component in the simplified box model has an important effect on the stability of the THC and can even generate new nondeterministic equilibria.

These findings motivate the idea that the determination of such critical noise levels in climate models is an important factor in assessing the reliability of future climate projections. All GCMs use parametrizations of subgrid-scale processes such as to account, for example, for the integral effect of clouds in the atmosphere or eddies in the ocean. Parameterizations can be regarded as functional relations that predict, in most cases, the mean values of the unresolved physical quantities rather than their complete statistics. Accounting also for the scatter around such functional relations in terms of stochastic forcing can be a natural application of the strategy pursued here: implementing multiplicative noise into the dynamical system equations. A similar ansatz was proposed by von Storch (1997). Apart from such an uncertainty analysis of climate models it can be argued that different climate mean states are associated with different statistics and vice versa. This is captured by the multiplicative noise ansatz used here. The external fluctuations affect the mean state of the system and the latter has a reciprocal effect on the intensity of the external forcing. During the Last Glacial period, for example, the atmospheric eddy statistics was considerably different from today (Hall et al. 1996). This might have been associated also with different stability properties of the THC. One might speculate that rapid transitions during the last glacial period such as the Dansgaard–Oeschger events could have been triggered by changes of the synoptic scale weather or short-term climate variability. A similar argumentation could also be applied to the greenhouse warming case.

The two-sided interaction between the short-term stochastic fluctuations and the slow dynamical component can be regarded as a metaphoric model for *timescale interactions.*

Stommel and Young (1993) have argued that, for fixed ΔT, $\langle y \rangle \sim 0.5$ is rather stable with respect to different additive freshwater noise levels. They suggest that the Stommel (1961) model provides a T/S regulation mechanism that could easily explain the observed value of $\langle y \rangle \sim 0.5$. The most likely value of y in our approach depends on the stochastic temperature noise in a very sensitive way, thus serving as a destabilizing factor for the mixed layer T/S regulation of Stommel and Young (1993).

Another issue raised here is the question of how reliable linear deterministic stability analysis is. It has become a very powerful tool in different climatological contexts. However, one should note that deterministic stationary states and their stochastic counterparts can differ strongly and even the number of stochastic equilibria might be different. Furthermore, we have shown that the linear deterministic stability analysis must be revised in the stochastic context as inferred from the Lyapunov exponents formula of the stochastic system and the application of the UCNA approach.

We think our analysis provides a good starting point for understanding the qualitative behaviour of other low order models in climate research. Overall our findings suggest that the statistics of unresolved physical processes is an important factor to understand the sensitivity and long-term dynamics of the climate system and should be analyzed in more complex climate models.

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