Automatica 144 (2022) 110487

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Sliding modes in the management of renewable resources[☆]

Anton Bondarev[a], Thorsten Upmann[b,c,d,*]

[a] International Business School Suzhou, Xi'an Jiaotong-Liverpool University, Ren'ai Road 111, 215123 Suzhou, PR China
[b] Helmholtz-Institute for Functional Marine Biodiversity at the University of Oldenburg (HIFMB), Ammerländer Heerstraße 231, 23129 Oldenburg, Germany
[c] Carl von Ossietzky Universität Oldenburg, Institute for Chemistry and Biology of the Marine Environment (ICBM), Germany
[d] Bielefeld University, Faculty of Business Administration and Economics, Bielefeld, Germany

ARTICLE INFO

Article history:
Received 9 July 2021
Received in revised form 21 May 2022
Accepted 5 June 2022
Available online 25 July 2022

Keywords:
Natural resources management
Optimal control of hybrid systems
Control of switched systems
Sliding mode control
Optimal control theory

ABSTRACT

The presence of tipping points in ecological systems implies abrupt changes in the dynamics of the ecosystem. In these piecewise-smooth dynamical systems sliding dynamics, *i.e.*, dynamics on the switching boundary, have been reported for population models. However, the question whether or not, and if so under which conditions, sliding dynamics may occur in an *optimally controlled system* have not yet been studied. We explore this issue in a simple harvesting model with two regimes, and find that optimal sliding may occur if regular steady states do not exist. Hence, sliding dynamics may be part of an optimal policy.

1. Introduction

Environmental economists have been paying a lot of attention to problems of optimal harvesting a renewable bio-resource,[1] while a parallel branch investigates dynamic harvesting games played by several players.[2] Models from both branches usually admit a smooth system, where the dynamics of the resource are described by a differential equation without jumps. At the same time, in the field of population dynamics there is a growing literature on applications of piecewise-smooth dynamical systems to the evolution of some species; recent examples are Tan, Qin, Liu, Yang, and Jiang (2016), Tang, Qin, and Tang (2014a), Tang, Tang, and Qin (2014b), Zhang and Tang (2013), among others. While these papers disregard optimal behaviour in any form, they show, applying Filippov's sliding flow (Filippov, 1988), that in standard dynamical systems, the presence of a threshold on the stock variable may lead to sliding dynamics: that is, the stock may evolve for some time along the threshold.

Two recent examples where the authors study sliding mode dynamics in a Filippov system are Bhattacharyya, Roelke, Pal, and Banerjee (2019) and Bhattacharyya, Piiroinen, and Banerjee (2021). These authors analyse predator–prey interactions of fish species in 2D and 3D dynamical systems, respectively, demonstrating the existence of sliding modes and the convergence of dynamics to the pseudo-equilibrium of the system. However, the harvesting policies in these papers follow simple, exogenously determined threshold strategies: once the resource stock exceeds a critical level, a given harvesting policy is implemented; otherwise, no harvesting takes place. That is, the specified a piecewise-continuous harvesting policy is not determined in an optimal way with respect to some criterion, and thus does not satisfy the Maximum Principle.

Hence, while sliding dynamics have been reported for many population systems, the conditions under which sliding dynamics may occur in an *optimally controlled system* have, as far as our knowledge extends, not yet been studied. In this paper, we aim to explore this issue and therefore combine both approaches: Applying methods known from hybrid control systems and the theory of *piecewise-smooth systems* (PWS system) to *optimally controlled* population dynamics, we explore the optimal harvesting strategy when the dynamics of a renewable natural resource

☆ The work of A. Bondarev was supported by the Key Program Special Fund of Xi'an Jiaotong-Liverpool University, China (grant no. KSF-E-63). The material in this paper was partially presented at None. This paper was recommended for publication in revised form by Associate Editor Kok Lay Teo under the direction of Editor Ian R. Petersen.

* Corresponding author at: Helmholtz-Institute for Functional Marine Biodiversity at the University of Oldenburg (HIFMB), Ammerländer Heerstraße 231, 23129 Oldenburg, Germany.

E-mail addresses: anton.bondarev@xjtlu.edu.cn (A. Bondarev), tupmann@hifmb.de (T. Upmann).

[1] Recent examples are Aniţa, Behringer, and Moşneagu (2019), Behringer and Upmann (2014), Belyakov, Davydov, and Veliov (2015), Belyakov and Veliov (2014), Cruz-Rivera, Ramirez, and Vasilieva (2019), Dubey, Agarwal, and Kumar (2018), Grass, Uecker, and Upmann (2019), Moberg, Pinsky, and Fenichel (2019), Upmann and Behringer (2020).

[2] Notable recent contributions are Dasgupta, Mitra, and Sorger (2019), de Frutos and Martin-Herran (2019), Fabbri, Faggian, and Freni (2020), Herrera, Moeller, and Neubert (2016), Mitra and Sorger (2014), among others.

https://doi.org/10.1016/j.automatica.2022.110487

abruptly change at some stock level (threshold). To this end, we deliberately adapt a standard single-species, optimal harvesting model, and extend the stock dynamics to a PWS system. This, compared to, for example, the predator–prey model of Bhattacharyya et al. (2021, 2019), rather simple framework allows us to derive closed form solutions for the optimal harvesting policies and to algebraically study the emergence of *optimal* sliding in a 2D dynamical system.

We demonstrate that sliding dynamics may be a constituent of an optimal harvesting policy, even in a simple harvesting model. Specifically, we demonstrate that in optimally controlled systems a Filippov's sliding flow can be the only feasible optimal outcome, if regular steady states are not feasible, a phenomenon to which we refer as *optimal sliding*; in this case, a novel steady state at the threshold level emerges, called *pseudo-equilibrium* in accordance with the literature on PWS systems (Colombo & Jeffrey, 2011; Di Bernardo, et al., 2008; Jacquemard, Teixeira, & Tonon, 2013), which is part of an optimal solution.

2. The theory of piecewise-smooth systems

For the case of a single switching condition, a PWS system for $\mathbf{x} \in \mathbb{R}^n$ may be defined as

$$\dot{\mathbf{x}} = \begin{cases} f_-(\mathbf{x}), & \text{if } \alpha(\mathbf{x}) < 0, \\ f_+(\mathbf{x}), & \text{if } \alpha(\mathbf{x}) > 0, \end{cases} \tag{1}$$

where $\alpha : \mathbb{R}^n \to \mathbb{R}$ and $f_- \neq f_+$. We refer to the vector fields f_- and f_+ as the *lower* and the *upper* vector fields. Following Filippov (1988), the solution of (1) at the switching manifold $\Sigma \equiv \{\mathbf{x} \mid \alpha(\mathbf{x}) = 0\}$ is formally defined as a differential inclusion

$$\dot{\mathbf{x}} \in f_\ell \equiv f_- + \varphi(f_+ - f_-) \tag{2}$$

where $\varphi = 0$ when $\alpha(\mathbf{x}) < 0$, $\varphi = 1$ when $\alpha(\mathbf{x}) > 0$, and $\varphi \in (0, 1)$ when $\alpha(\mathbf{x}) = 0$. The *flow* through a point $\hat{\mathbf{x}}$ at time t is given by all points $\mathbf{x}(t+\tau)$ with $\mathbf{x}(\tau) = \hat{\mathbf{x}}$ for some $\tau \in \mathbb{R}$ and $\mathbf{x}(t)$ satisfying (2).

In general, the switching manifold Σ is an $n-1$ dimensional manifold, and we henceforth denote the generic element of Σ, *i.e.* the generic root of $\alpha(\mathbf{x}) = 0$, by $\mathbf{x}_s$. Correspondingly, the (generic) steady states of f_- and f_+ are denoted by $\mathbf{x}_-^\infty$ and $\mathbf{x}_+^\infty$. Either f_- or f_+ may have unique or multiple steady states; these steady states can be classified as follows.

Definition 1. A steady state $\mathbf{x}_+^\infty$ of an upper vector field is called (i) *regular* if $\alpha(\mathbf{x}_+^\infty) > 0$, (ii) *virtual* if $\alpha(\mathbf{x}_+^\infty) < 0$, (iii) *boundary* if $\alpha(\mathbf{x}_+^\infty) = 0$. Similarly, a steady state $\mathbf{x}_-^\infty$ of a lower vector field is called (i) *regular* if $\alpha(\mathbf{x}_-^\infty) < 0$, (ii) *virtual* if $\alpha(\mathbf{x}_-^\infty) > 0$, and (iii) *boundary* if $\alpha(\mathbf{x}_-^\infty) = 0$.

We denote by $\mathcal{L}_f\alpha \equiv \langle f, \nabla\alpha \rangle$ the Lie derivative of α along the vector field f. The topology of the switching manifold Σ consists of three regions (see, *e.g.* Jacquemard et al., 2013):

Definition 2. The disjoint subsets Σ_{CR}, Σ_{ES}, Σ_{SL} of the switching manifold Σ are called

- the *crossing region*: $\Sigma_{CR} \equiv \{\mathbf{x} \in \Sigma : (\mathcal{L}_{f_+}\alpha)(\mathbf{x})(\mathcal{L}_{f_-}\alpha)(\mathbf{x}) > 0\}$, where the trajectory may cross the switching manifold from one vector field to the other;
- the *escaping region*: $\Sigma_{ES} \equiv \{\mathbf{x} \in \Sigma : (\mathcal{L}_{f_+}\alpha)(\mathbf{x}) > 0, (\mathcal{L}_{f_-}\alpha)(\mathbf{x}) < 0\}$, where both vector fields are bounced off the switching manifold;
- the *sliding region*: $\Sigma_{SL} \equiv \{\mathbf{x} \in \Sigma : (\mathcal{L}_{f_+}\alpha)(\mathbf{x}) < 0, (\mathcal{L}_{f_-}\alpha)(\mathbf{x}) > 0\}$, where both vector fields point into the switching manifold.

If $\Sigma_{SL} \neq \emptyset$, the sliding vector field may be defined on Σ_{SL} specifying the dynamics on this part of the switching manifold. This can be done via Utkin's method (see Utkin, 1992) or Filippov's method (see Filippov, 1988). When f_- and f_+ have opposite directions at Σ (escaping and/or sliding regions exist), there exists a solution that lies on the switching manifold and satisfies $\dot{\mathbf{x}} = f_\ell(\mathbf{x})$ where f_ℓ is a *sliding vector field*:

$$f_\ell \equiv f_- + \frac{\mathcal{L}_{f_-}\alpha}{\mathcal{L}_{f_-}\alpha - \mathcal{L}_{f_+}\alpha}(f_+ - f_-). \tag{3}$$

The dynamics at $\dot{x} \equiv f_\ell$ is also known as *Filippov's flow* (see Colombo & Jeffrey, 2011), which possesses its own steady states, presuming their existence, $\mathbf{x}_\ell^\infty : f_\ell(\mathbf{x}_\ell^\infty) = 0$. This type of a steady state is called a *pseudo-equilibrium* of the PWS system (1) since $\mathbf{x}_\ell^\infty$ is neither a steady state of the field f_- nor of f_+. Still, a steady state $\mathbf{x}_\ell^\infty$ can be stable or unstable, based on the properties of the sliding vector field f_ℓ alone.

3. A simple harvesting model

We start with a basic optimal harvesting model which we then extend to an optimal control (OC) problem with PWS dynamics. Assume an agent who seeks to maximise the utility stream from harvesting a renewable natural resource net of the associated harvesting cost, over an infinite time horizon $\mathcal{T} \equiv [0, \infty)$. Let $x \in \mathbb{R}_+$ denote the stock (or abundance) of the resource, and $u \in \mathcal{U} \subset \mathbb{R}_+$ the harvesting effort. We assume that $t \mapsto x(t)$ is absolutely continuous, $t \mapsto u(t)$ is piecewise continuous, and that $\mathcal{U}$ is compact.

The growth of the stock is governed by the logistic growth function $rx\left(1 - \frac{x}{K}\right)$, where $r > 0$ and $K > 0$ denote the growth rate and the carrying capacity of the stock. We specify the harvesting yield as a bilinear function of the harvesting effort and the stock: $h(x, u) = qxu$; here q is the catchability coefficient, defined as the fraction of the stock harvested per unit of effort u. For convenience, we normalise the units of effort to set $q = 1$, so that $h(x, u) = xu$. Then, the resulting growth process of the stock is governed by

$$\dot{x}(t) = f(x(t), u(t)) \equiv x(t)\left[r\left(1 - \frac{x(t)}{K}\right) - u(t)\right], \tag{4a}$$

for all $t \in \mathcal{T}$ and $x(0) = x_0$. For constant u, there are two steady state stocks: 0 and $K\left(1 - \frac{u}{r}\right)$, the latter being positive if the growth rate exceeds the constant depletion rate, *i.e.* if $r > u$. Hence, the steady state harvesting rate is bounded by the natural growth rate of the stock.

The instantaneous payoff of the agent equals revenue minus the associated cost of harvesting. Specifically, let p be the (constant) price of one unit of the harvested resource, then the revenue of the harvest is $ph(x, u) = pxu$. Moreover, the associated harvesting costs are assumed to be quadratic in the catch, $(c/2)h(x, u)^2 = (c/2)(xu)^2$, with $0 \leq c < p$. Then, instantaneous profit of the agent amounts to

$$J_c(x(t), u(t)) \equiv px(t)u(t) - \frac{1}{2}cu(t)^2x(t)^2, \quad \forall t \in \mathcal{T}$$

and the problem for the agent becomes to maximise the (discounted) profit stream for the planning period $\mathcal{T}$, subject to the growth of the stock:

$$\max_{u(\bullet)} J \equiv \int_0^\infty e^{-\rho t} J_c\left(x(t), u(t)\right)\, dt, \tag{4b}$$

subject to (4a). where $\rho > 0$ denotes the discount rate. Applying Pontryagin's maximum principle yields the optimal control $u^*(x(t), \lambda(t)) \equiv (p - \lambda(t))/(cx(t))$. Inserting u^* into the state and the costate equation gives, together with the initial condition

$x(0) = x_0$ and the transversality condition $\lim_{t\to\infty} \mathrm{e}^{-\rho t}\lambda(t) = 0$, the *canonical system* (CS). Using the notation $X \equiv (x, \lambda) \in \mathcal{X} \equiv \mathbb{R}_+ \times \mathbb{R}$, the CS becomes

$$\dot{X} \equiv \begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} \equiv \begin{pmatrix} \frac{(\lambda(t)-p)}{c} + \frac{rx(t)(K-x(t))}{K} \\ \lambda(t)\left(r\left(\frac{2x(t)}{K} - 1\right) + \rho\right) \end{pmatrix}, \tag{5}$$

which we henceforth compactly write as $\dot{X} = G(X)$. Assuming that (5) is a continuous differential equation system, it represents a smooth ODE system, *i.e.* $G : \mathcal{X} \to \mathcal{X} : X \mapsto G(X)$ is a differentiable function which we want to solve over the planning horizon $\mathcal{T}$. Yet, (5) is not an initial value problem, as it provides initial data only for the state variable, but gives a terminal condition, *viz.* the transversality condition for the costate variable. This becomes a numerical issue when we compute the *canonical path* (CP) $t \mapsto X(t) \equiv (x(t), \lambda(t))$ connecting a given initial state x_0 to some steady state $X^\infty \equiv (x^\infty, \lambda^\infty)$ of the CS (5), to which we refer as *canonical steady state* (CSS). (We next show that (5) possesses a CSS.) Such a path from some x_0 to a CSS X^∞ exists only if X^∞ has the *saddlepoint property* (SPP), which may or may not be the case.[3]

We begin with the calculation of the CSSs and an analysis of their stability. Here, the unique real valued (non-trivial) CSS $X^\infty = (x^\infty, \lambda^\infty)$ of (5) is

$$X^\infty = \left(\frac{K(r-\rho)}{2r},\ p + \frac{cK}{4r}\left(\rho^2 - r^2\right)\right)$$

and an associated optimal steady-state control $u^\infty = (\rho + r)/2$. In order to have an economically meaningful model, we henceforth presuppose

Assumption 3. Let $r > 0$, $\rho > 0$ and $0 < r^2 - \rho^2 < (4pr)/(cK)$, which, in particular, implies $r > \rho > 0$.

Intuitively, Assumption 3 says that (i) the growth rate must exceed the discount rate (or that future is not discounted too strongly compared to the growth of the stock; (ii) the (market) price of the resource must be high enough compared to the harvesting cost. If so, then under Assumption 3, the steady state is positive, *i.e.* $X^\infty \in \mathbb{R}^2_{++}$. More generally, we shall limit both the state and the costate variable to be non-negative, *i.e.* $X(t) \in \mathcal{X} \equiv \mathbb{R}^2_+$. While negative values of the state variable are automatically excluded by the formulation of the model, negative costate variables are excluded because they make little sense economically.

It is straightforward to show that the Jacobian of (5) has two real eigenvalues, with the larger one being positive and the lower one, under Assumption 3, being negative. Hence, under Assumption 3, X^∞ is a saddle point of the CS (5), with the stable manifold $W_s(X^\infty)$ approaching X^∞ from the north-west and from the south-east. Since X^∞ is the unique CSS and it has the SPP, it is also globally optimal. That is, given some x_0, the optimal solution is unique and converges to X^∞.

4. Optimal harvesting with PWS dynamics

We now consider the harvesting problem when the growth rate of the stock r experiences an upward (downward) jump when the stock passes some fixed threshold level $x_s \in (0, K)$. While the growth rate experiences a discrete change, the stock variable, though, does not experience any discontinuity at x_s. This model describes an ecological situation where the growth rate of the species is endogenous, depending on the stock size. A primary example is a fish stock the growth rate of which rises or falls once some critical level of abundance is reached. A rise describes situation of exploding growth when external resources are abundant and do not limit the population; a fall, when over-population of fish leads to severe competition for scarce resources resulting in strong overcrowding effects. Other examples may include pollution management (with runaway climate change being the motivating idea), forest resources, and more broadly any renewable resource management model.

Let the subscripts minus (-) and plus (+) refer to the two regimes — the lower and the upper regime. Then the corresponding state dynamics can be written as

$$\dot{x}(t) = \begin{cases} f_-(x(t), u(t)) & \text{if } \alpha(x(t)) < 0, \\ f_+(x(t), u(t)) & \text{if } \alpha(x(t)) > 0. \end{cases} \tag{6a}$$

where $f_\pm \equiv r_\pm x(t)\left(1 - \frac{x(t)}{K}\right) - x(t)u(t)$ and $\alpha(x) \equiv x - x_s$. Here, $\alpha(x) = 0$ represents the *switching condition*, and $\Sigma \equiv \{x \in \mathbb{R}_+ : \alpha(x) = 0\}$ the corresponding *switching manifold* of system (6a). With the stock dynamics (6a), the problem of the agent becomes

$$\max_{u(\bullet)} \int_0^\infty \mathrm{e}^{-\rho t} J_c(x(t), u(t))\, \mathrm{d}t, \ \text{ s. t. (6a)}. \tag{6b}$$

To derive a solution for problem (6), we need to construct a Hamiltonian for each regime and to specify an appropriate jump condition for the costate variable on Σ. So, we define the Hamiltonian functions $H_\pm(x, u, \lambda) \equiv J_c(x, u) + \lambda f_\pm(x, u)$, and apply the standard maximum principle to each of them. Using the optimality conditions $\partial H_\pm(x, u^*_\pm, \lambda)/\partial u = 0$, we obtain the optimal controls $u^*_\pm = u^*(t) = (p - \lambda(t))/(cx(t))$. Substituting these into the corresponding state and the costate equations, we arrive at the two corresponding CSs, *viz.* CS_+ and CS_-:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{\lambda}(t) \end{pmatrix} = \begin{pmatrix} \frac{(\lambda(t)-p)}{c} + \frac{r_\pm x(t)(K-x(t))}{K} \\ \lambda(t)\left(r_\pm\left(\frac{2x(t)}{K} - 1\right) + \rho\right) \end{pmatrix}. \tag{7}$$

We henceforth write (7) more compactly as $\dot{X} = G_\pm(X) := (f_\pm, g_\pm)$, where, with minor sloppiness, we write $f_\pm$ for $f_\pm(x, u^*)$. We then refer to $\dot{X} = G_-(X)$ as the vector field of the *lower regime*, and to $\dot{X} = G_+(X)$ as the vector field of the *upper regime* of (7) – and the switching condition $\alpha(x(t)) = 0$ defines which of these systems applies. We then have the CS of the original problem described by the PWS system

$$\dot{X}(t) = \begin{cases} G_-(X(t)) & \text{if } \alpha(x(t)) < 0, \\ G_+(X(t)) & \text{if } \alpha(x(t)) > 0, \end{cases} \tag{8}$$

together with the transversality condition $\lim_{t\to\infty} \mathrm{e}^{-\rho t}\ \lambda(t) = 0$. The transversality condition stays the same as for the standard model (see Shaikh & Caines, 2007), but an additional condition at each time there is a jump between the two stages is required. Let t_s denote the (generic) time when the switching manifold is reached. Then, the matching conditions for the optimal switching time t^*_s are (see Bondarev & Gromov, 2021, for details):

$$\lambda_1(t^*_s) = \frac{\partial J_1}{\partial x}(t^*_s, x_s) = \lambda_2(t^*_s), \tag{9a}$$

$$H_1(t^*_s) = \frac{\partial J_1}{\partial t_s}(t^*_s, x_s) = H_2(t^*_s). \tag{9b}$$

where $J_1(0, x_0) \equiv \int_0^{t_s} \mathrm{e}^{-\rho t} J_c(x(t), u(t))\mathrm{d}t + J_1(t_s, x_s)$ is the value of the objective functional in one of the regimes with $J_2(t_s, x_s) \equiv \int_{t_s}^\infty \mathrm{e}^{-\rho t} J_c(x(t), u(t))\mathrm{d}t$ being the *terminal cost* entering that functional from the other regime; and an analogous condition for λ_2

[3] In general, a CSS $X^\infty \in \mathbb{R}^{2n}$ has the SPP if the dimension of the stable manifold equals the dimension of the unstable manifold of X^∞, *i.e.* $\dim W_s(X^\infty) = \dim W_u(X^\infty) = n$. The number $d(X^\infty) \equiv n - \dim W_s(X^\infty)$ is called the *defect* of the CSS X^∞; and a CSS X^∞ with $d(X^\infty) > 0$ is called *defective*, whereas a CSS X^∞ with $d(X^\infty) = 0$ has the SPP. For more details on the SPP for OC problems see, for example, Grass, Caulkins, Feichtinger, Tragler, and Behrens (2008, p. 238) and Upmann, Uecker, Hammann, and Blasius (2021, p. 4f).

and H_2. This condition requires that the opportunity cost (costate value) before reaching the switching condition equals the marginal revenue accumulated in the following dynamic in the upper regime. We thus obtain *two* CSs: one for each regime, and the solution of control problem (6) consists of two parts: the solution of the lower and of the upper regime. The two solutions are connected via the continuity of the state variable(s) at the switching manifold Σ.

The unique CSS for each regime, *i.e.* the root of $G_\pm(X)$, denoted by $X_\pm^\infty = \left(x_\pm^\infty, \lambda_\pm^\infty\right)$, is given by

$$X_\pm^\infty = \left(\frac{K(r_\pm - \rho)}{2r_\pm},\ p - \frac{cK}{4r_\pm}\left(r_\pm^2 - \rho^2\right)\right), \tag{10}$$

and the associated optimal control by $u_\pm^\infty = (\rho + r_\pm)/2$, yielding the steady state profit rate

$$J_c^\infty = -\frac{K}{32r_\pm^2}(r_\pm - \rho)(r_\pm + \rho)\left(8pr_\pm - cK\left(r_\pm^2 - \rho^2\right)\right).$$

Assumption 3 is a sufficient condition for J_c^∞ to be positive; also, $\partial J_c^\infty/\partial r$ is positive if, and only if, Assumption 3 holds.

Lemma 4. *Consider the hybrid system (8).*

(1) If $r_- < r_+$, then $x_-^\infty < x_+^\infty$ and at least one of the CSS is regular, and sliding dynamics of type (3) cannot emerge. In particular:

- *(i) If $x_s < x_-^\infty$, the unique CSS is X_+^∞;*
- *(ii) If $x_s > x_+^\infty$, the unique CSS is X_-^∞;*
- *(iii) If $x_-^\infty < x_s < x_+^\infty$, there are two CSS, X_-^∞ and X_+^∞.*

(2) If $r_- > r_+$, then $x_-^\infty > x_+^\infty$, and sliding dynamics may emerge. In particular:

- *(i) If $x_s < x_+^\infty$, the unique CSS is X_+^∞;*
- *(ii) If $x_s > x_-^\infty$, the unique CSS is X_-^∞;*
- *(iii) If $x_+^\infty < x_s < x_-^\infty$, none of the $X_\pm^\infty$ are feasible.*

The proof of Lemma 4 is, as are the proofs of all other results, relegated to the Appendix.

Fig. 1 illustrates the case of Lemma 4 part 1, *i.e.* the situation $r_- < r_+$. Therein, the left and the right diagrams illustrate the cases of a unique saddle-type CSS in the upper and in lower regime, respectively. In either of the two cases, any optimal trajectory will converge to the unique steady state. More precisely, if $x_s < x_-^\infty < x_+^\infty$, see diagram Fig. 1(a), the optimal path is to follow the stable saddle path of X_+^∞, unless $x(t) < x_s$ where we have to follow the stable saddle path (red path) of X_-^∞ until x_s is reached. Upon arrival at x_s at time τ, *i.e.*, when $x(\tau) = x_s$, the costate variable experiences an upward-jump from $\lambda(\tau^-)$ to $\lambda(\tau^+)$. In the case when $x_-^\infty < x_+^\infty < x_s$, see diagram Fig. 1(c), the optimal path is analogous: it coincides with the stable saddle path of X_-^∞, unless $x(t) > x_s$ where we have to follow the stable saddle path (red path) of X_+^∞ until x_s is reached. Here, the costate variable jumps upwards from $\lambda(\tau^+)$ to $\lambda(\tau^-)$. In the centre diagram Fig. 1(b), though, there exist two saddle-type CSS, one for the lower and one for the upper regime, and depending on x_0, either of them may be reached by following the stable saddle path either of X_-^∞ if $x(t) < x_s$, or of X_+^∞ if $x(t) > x_s$. In all three cases, the optimal solution is obtained via a suitable version of the maximum principle (standard or hybrid), and the optimal path is given by the stable manifold of X_-^∞ if $x(t) < x_s$ and of X_+^∞ if $x(t) > x_s$, with a jump in the costate upon arrival at x_s.

Fig. 2 illustrates Lemma 4 part 2, *i.e.* the situation $r_- > r_+$, implying $x_+^\infty < x_-^\infty$. The situation when the switching level of the stock is low, such that $x_s < x_+^\infty$, is illustrated in diagram Fig. 2(a), with $x_s = 0.2$; in this case, only X_+^∞, indicated by the

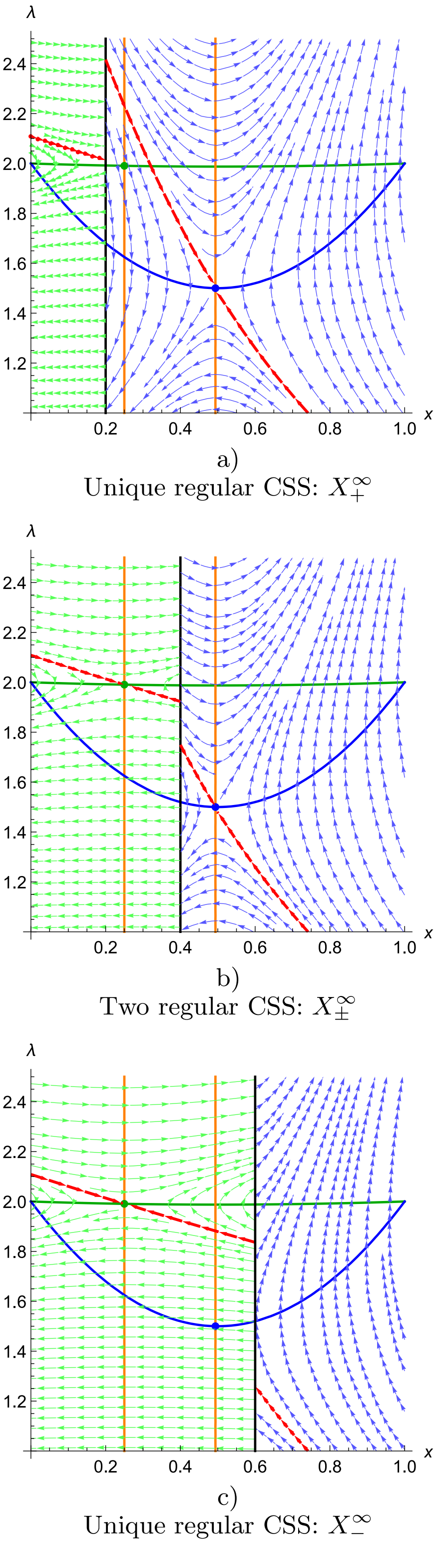

Fig. 1. Phase diagram for the case $r_+ = 2 > 0.05 = r_-$, with $x_s = 0.2$ (left) and $x_s = 0.4$ (centre) and $x_s = 0.6$ (right). The stable saddle paths (in red) represent the optimal path for the respective region of the state variable.

blue point, is a regular CSS. The case when the switching level of the stock is high, such that $x_s > x_-^\infty$, is illustrated in diagram Fig. 2(c), with $x_s = 0.6$; in this case, only X_-^∞, indicated by the red point, is a regular CSS. As before, in both cases the optimal path is given by the stable manifold of X_-^∞ if $x(t) < x_s$, and of

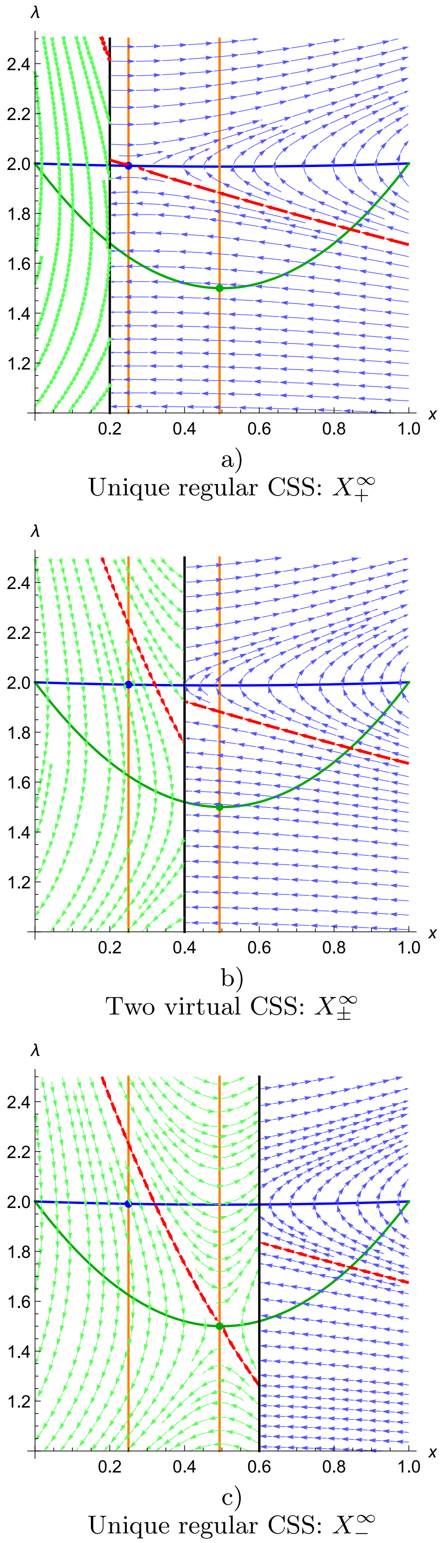

Fig. 2. Phase diagram for the case $r_+ = 0.05 < 2 = r_-$, with $x_s = 0.2$ (left) and $x_s = 0.4$ (centre) and $x_s = 0.6$ (right).

X_+^∞ if $x(t) > x_s$, with a jump in the costate upon arrival at x_s. In the intermediate case, though, where the switching level lies between the two steady state stocks, *i.e.* $x_+^\infty < x_s < x_-^\infty$, both X_-^∞ and X_+^∞ are virtual, so that neither of them is feasible. This is illustrated in diagram Fig. 2(b) with $x_s = 0.4$; it shows why sliding dynamics may be an optimal outcome: With both CSS being virtual, there is no trajectory leading to either of them; in particular, there is no trajectory crossing the manifold that leads to a CSS. However, there exists a region in the neighbourhood of the switching manifold where $G_-(X)_{x=x_s} > 0$ and $G_+(X)|_{x=x_s} < 0$, indicating the existence of sliding dynamics as in Definition 2.[4] And since neither of the CSS can be reached, the only candidate for an optimal path is to follow the stable manifold of either X_-^∞, if $x(t) < x_s$, or X_+^∞, if $x(t) > x_s$, entering the switching manifold in finite time. We will now elaborate on the optimality of sliding in more detail.

5. Optimal sliding

Since the sliding surface is reachable for $r_+ < r_-$, we have to define the dynamics along the switching manifold Σ. Following Filippov (1988), the flow on Σ is a linear combination of the flows from both sides evaluated at the threshold, the codimension of which equals $2n - 1 = 1$ as we have $n = 1$. Applying formula (3) and taking into account that Σ does not depend on λ but is vertical line at x_s, we obtain a 1D flow of the costate variable

$$\dot{\lambda} \equiv \left[g_- + \frac{f_-}{f_- - f_+} (g_+ - g_-) \right]_{x=x_s}, \tag{11}$$

where $G_- = (f_-, g_-)$ and $G_+ = (f_+, g_+)$, as defined in (8). Since the sliding flow happens at x_s, the crossing, the escaping and the sliding regions are intervals of the line $(x_s, \cdot)$. To emphasise this, we denote these intervals by Λ_{CR}, Λ_{SL} and Λ_{ES}. Specifically, the sliding region Σ_{SL} is given by all pairs (x_s, λ) with $\lambda \in \Lambda_{SL} \equiv (\lambda_{min}, \lambda_{max})$, where λ_{min} and λ_{max} define the lower and the upper boundary of the sliding region. That is, the interval Λ_{SL} representing the general sliding region Σ_{SL} is defined from the geometry of both vector fields G_- and G_+, whereas points (x_s, λ) with $\lambda \notin \Lambda_{SL}$ lie outside the sliding region.

If there is an optimal trajectory leading to the sliding region Σ_{SL}, then it is unique due to the geometry of the sliding region, where both the lower and the upper vector fields are pointing to Σ, *i.e.*, $G_-(X)_{x=x_s} > 0$ and $G_+(X)|_{x=x_s} < 0$. Once a trajectory has entered the sliding region Σ_{SL}, the dynamics are governed by the 1D sliding flow λ_ℓ, where x is fixed at x_s. Eventually though, the sliding solution λ_ℓ reaches the boundary of Λ_{SL}, and the trajectory exits Λ_{SL}; and upon exit, the trajectory enters the vector field of either the lower or the upper regime, depending on the particular geometry of both vector fields in the neighbourhood of the respective exit point (x_s, λ_{min}) or (x_s, λ_{max}) — and it will never enter the sliding region Σ_{SL} again. As a consequence, if the optimal trajectory enters the sliding region, and therefore sliding is part of the optimal solution, this entry is unique and the sliding mode is optimal only for a single time interval $[\tau_1, \tau_2] \in \mathcal{T}$, unless the optimal trajectory enters the sliding region at the steady state of the sliding vector field, *i.e.* either $\lambda_-(\tau_1) = \lambda_\ell^\infty$ or $\lambda_+(\tau_1) = \lambda_\ell^\infty$.

Since the 2D sliding dynamics $\dot{X} = G_\ell(X)$ are defined for a fixed value of the first variable, $x = x_s$, it suffices to define these dynamics in one dimension: $\dot{\lambda}(t) = g_\ell(\lambda)$, where g_ℓ is the second component of G_ℓ:

$$\dot{\lambda}(t) = g_\ell (\lambda(t)) = \lambda(t)\rho + \lambda(t) \frac{(\lambda(t) - p)}{c x_s \psi}, \tag{12}$$

for all $\lambda \in \Lambda_{SL}$, with $\psi \equiv (K - x_s)/(K - 2x_s)$. We denote the (general) solution of this ODE by λ_ℓ, referring to it as the *sliding flow*. While the sliding flow of the costate variable is given by (12),

[4] In our case $\alpha(X) = x - x_s$, so the general definition $\{\alpha(X) = 0 : (\mathcal{L}_{G_+}\alpha)(X) < 0, (\mathcal{L}_{G_-}\alpha)(X) > 0\}$ reduces to $G_-(x_s, \lambda)\nabla\alpha(x_s) > 0$, $G_+(x_s, \lambda)\nabla\alpha(x_s) < 0$ with $\nabla\alpha(x_s) = \{1, 0\}$, simply yielding $G_-(x_s, \lambda) > 0$, $G_+(x_s, \lambda) < 0$ for any λ, or more compactly, $G_-(X)_{x=x_s} > 0$, $G_+(X)|_{x=x_s} < 0$.

the stock variable is fixed at x_s. Taking the time derivative of u^* and evaluating terms at x_s, the flow of the optimal control u^* along the switching manifold is given by

$$\dot{u}^*(t) = -\frac{a}{cx_s}\dot{\lambda}(t) = -\frac{a}{cx_s}g_\ell(\lambda(t)). \tag{13}$$

Hence, during times of sliding, the change in u^* is linearly but negatively related to the change in λ: Any increase (decrease) in λ implies decrease (increase) in u^* leaving stock at the constant level x_s. Now we study whether such sliding control can be optimal. To this end, we make the following

Definition 5. A trajectory of CS (8) exhibits *optimal sliding* if there exists an interval $[\tau_1, \tau_2] \subset \mathcal{T}$ such that $\forall t \in [\tau_1, \tau_2]$: $\dot{X}(t) = G_\ell$ and this trajectory maximises the objective payoff of (6b) among the admissible ones.

Since the right hand side of (12) is quadratic in $\lambda(t)$, this equation has at most two regular steady states:

$$\lambda_\ell^\infty \in \{0,\ p - \rho c x_s \psi\} \tag{14}$$

for $x_s \neq K/2$. While the first root is trivial and of no economic value (and hence is subsequently ignored), only the second is economically meaningful, provided it is positive. To this end, we henceforth assume that the market price of the harvested resource is sufficiently large:

Assumption 6. Let $p > \rho c x_s \psi$ and $x_s \neq K/2$.

Assumption 6 says that for $x_s < K/2$, there is a lower bound on p, increasing in x_s, for the resource to be valuable, *i.e.* $\lambda > 0$, at the steady state of the sliding flow. (For $x_s > K/2$ this bound is negative and thus irrelevant.) Now, the non-trivial root λ_ℓ^∞ may be feasible or infeasible, depending on the values of the parameters.

Definition 7. A point $X_\ell^\infty \equiv \left(x_s, \lambda_\ell^\infty\right)$, where λ_ℓ^∞ is the steady state of the sliding vector field g_ℓ, is called a *pseudo-equilibrium* of the PWS system (8).

Corollary 8. *There exists a unique non-trivial pseudo-equilibrium for (8), which is feasible if $\lambda_\ell^\infty \in \Lambda_{SL}$, and infeasible (i.e. virtual) otherwise.*

Thus, for a sliding motion to be a candidate for optimal dynamics, both (original) CSSs, *i.e.* X_-^∞ and X_+^∞, must be virtual, and the pseudo-equilibrium X_ℓ^∞ must be feasible. The latter requirement has two versions, which we provide as a formal definition:

Definition 9. The pseudo-equilibrium X_ℓ^∞ is

1. *weakly feasible* if $\lambda_\ell^\infty \in \Lambda_{SL}$ is an unstable steady state of g_ℓ and the pseudo-equilibrium X_ℓ^∞ has the SPP;
2. *strongly feasible* if $\lambda_\ell^\infty \in \Lambda_{SL}$ is a locally stable steady state of g_ℓ and the pseudo-equilibrium X_ℓ^∞ has the SPP;
3. *infeasible* if $\lambda_\ell^\infty \notin \Lambda_{SL}$ (*i.e.* the steady state of g_ℓ is virtual).

In the case of weak feasibility, the pseudo-equilibrium X_ℓ^∞ of the 2D PWS system is feasible only from the outside of the sliding region Σ_{SL}. More precisely, X_ℓ^∞ can be reached from a 1D manifold, and thus has the SPP. In the case of strong feasibility, though, X_ℓ^∞ can be reached from a stable 1D manifold lying outside of Σ_{SL}, and from Σ_{SL}. Hence, strong feasibility allows X_ℓ^∞ to be reached from two stable 1D manifolds: one that is unbounded and lies outside of Σ, and another one that is bounded and coincides with Σ_{SL}.

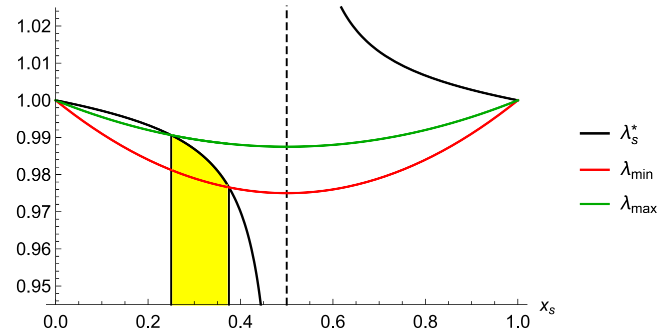

Fig. 3. Boundaries of the sliding domain (in yellow), for $K = 1$, $p = 1$, $c = 1/2$, $\rho = 0.05$, $r_- = 0.2$, $r_+ = 0.1$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Lemma 10. *Assume that the CSS of (8) satisfies $x_-^\infty > x_s > x_+^\infty$. A necessary condition for the sliding flow g_ℓ to be part of the optimal solution (optimal trajectory) is that $\lambda_\ell^\infty \in \Lambda_{SL}$.*

In order to establish the feasibility of a pseudo-equilibrium, we first define the bounds of the sliding region Λ_{SL}, and then explore the stability of the pseudo-equilibrium. The bounds of the sliding region can be found via Definition 2. For the CS (8), these bounds are given by a single inequality for λ:

Lemma 11. *For the CS (8), the set $\Lambda_{SL} \equiv (\lambda_{min}, \lambda_{max})$ is defined by all values of λ satisfying*

$$\lambda_{min} \equiv p - \gamma_s(r_-) \leq \lambda \leq p - \gamma_s(r_+) \equiv \lambda_{max}. \tag{15}$$

where $\gamma_s(r) := (crx_s(K - x_s))/K$.

Inequality (15) shows that for $r_+ < r_-$, the sliding region is non-empty provided that both sides of inequality (15) are non-negative. Now we can check whether or not the condition

$$\lambda_{min} \leq \lambda_\ell^\infty \leq \lambda_{max}, \tag{16}$$

holds. Specifically, this gives the following.

Lemma 12. *When x_s converges to zero, the sliding region Λ_{SL} vanishes, and we have $\lambda_{min} = \lambda_{max} = p$. Hence, the trivial pseudo-equilibrium $(x_s, 0)$ is never feasible.*

Since (16) represents a parametric condition in x_s, we can provide a condition for the switching level x_s such that sliding emerges. To this end, define $\kappa(r) \equiv \frac{K}{2}\frac{(r-\rho)}{r}$, with $\kappa(r) \in (0, K/2)$ since $r > \rho$, and $\kappa'(r) > 0$.

Lemma 13. *Let $r_- > r_+ > \rho$. The non-trivial pseudo-equilibrium X_ℓ^∞ is feasible (for $0 < x_s < K$) if, and only if, $\kappa(r_+) < x_s < \kappa(r_-)$.*

Fig. 3 illustrates Lemmas 11 and 13. Setting the parameters $K = 1, p = 1, c = 1/2, \rho = 0.05, r_- = 0.2, r_+ = 0.1$, we find that for all $x_s \in (0.25, 0.375)$ the condition $\lambda_{min} < \lambda_\ell^\infty < \lambda_{max}$ is satisfied. Hence, there is a non-vanishing interval of switching values for which sliding emerges (see yellow marked interval in Fig. 3).

Since $\kappa(r) \in (0, K/2)$, and thus $\kappa(r_-) < K/2$, sliding emerges if the switching level x_s is relatively low, *viz.* below half of the carrying capacity K. Intuitively, sliding only emerges if (i) the growth rate of the stock is lower for higher than for lower abundances, and if (ii) this drop of the growth rate emerges at relatively low abundances. Only if this drop emerges at a low stock level, is the growth path of the resource substantially impaired, so that the agent chooses a “precautionary” harvesting policy by maintaining the stock exactly at the switching level. If, however, the growth rate drops at abundances close to the carrying capacity, such a precautionary harvesting policy is not required, since the opportunity cost of the foregone growth (in absolute terms) is rather low.

The stability of a pseudo-equilibrium is defined by the sign of the derivative of the flow w. r. t. its variables at the steady state. In our case, this is a scalar, since the flow is 1D. The linearisation of (12), viz the scalar "Jacobian", evaluated at λ_ℓ^∞ equals

$$\frac{\mathrm{d}g_\ell}{\mathrm{d}\lambda}(\lambda_\ell^\infty) = \frac{p}{cx\psi} - \rho. \tag{17}$$

Under Assumption 6, this derivative is non-negative, and hence we obtain the following lemma.

Lemma 14. *Let $x_s \in (\kappa(r_+), \kappa(r_-))$. The pseudo-equilibrium X_ℓ^∞ is weakly (strongly) feasible if, and only if, Assumption 6 holds (does not hold).*

We know from Lemma 14 that, under Assumption 6, a pseudo-equilibrium X_ℓ^∞ is weakly feasible but not strongly feasible, provided that X_ℓ^∞ exists. Let $\mathcal{X}_{\mathrm{F}}$ denote the set of feasible steady states, $\mathcal{X}_{\mathrm{WF}}$ the set of weakly feasible steady states, and $\mathcal{X}_{\mathrm{SF}}$ the set of strongly feasible steady states, with $\mathcal{X}_{\mathrm{F}} \equiv \mathcal{X}_{\mathrm{WF}} \cup \mathcal{X}_{\mathrm{SF}}$ and $\mathcal{X}_{\mathrm{WF}} \cap \mathcal{X}_{\mathrm{SF}} = \emptyset$. Now, by Lemma 13 we know that $\lambda_\ell^* \in \mathcal{X}_{\mathrm{F}}$ if, and only if, $x_s \in (\kappa(r_+), \kappa(r_-))$.

Corollary 15. *Let Assumption 6 hold. The pseudo-equilibrium X_ℓ^∞ is weakly feasible but not strongly feasible, i.e. $\mathcal{X}_{\mathrm{F}} \equiv \mathcal{X}_{\mathrm{WF}}$ and thus $\mathcal{X}_{\mathrm{SF}} = \emptyset$.*

So, the only possibility to reach the pseudo-equilibrium is from the outside of the sliding region. Specifically, the sliding flow (12), where the co-state varies while the stock remains constant, can only be a part of an optimal solution when the pseudo-equilibrium is reached via the associated 1D manifold from either the upper or the lower regime, but not from the sliding flow itself. Moreover, these are the only trajectories that do not lead to either zero or infinity in finite time, so only these are candidates for a solution. Lastly, for the case of weak feasibility, we get uniqueness of the solution by replacing the standard transversality condition by the following one:

Lemma 16. *If the pseudo-equilibrium X_ℓ^∞ is weakly feasible and both steady states are virtual (i.e. steady state stock values are $x_-^\infty > x_s > x_+^\infty$), then:*

1. *The standard and hybrid maximum principles do not yield an optimal control;*
2. *The control associated with $CS_\pm$, adjoined by the condition*

$$\lim_{t\to t_s} \lambda(t) = \lambda_\ell^\infty \tag{18}$$

is the only optimal one, with $t_s \equiv \min_t\{t \mid x_\pm(t) = x_s\}$ being the time of first contact of $x_\pm(t)$ with the switching manifold.

Remark. The condition (18) is sufficient and necessary for both the upper and the lower regime, depending on the location of x_0. It requires the costate to approach a given *constant* value, since by (14) the pseudo-equilibrium value is defined for the given x_s.

In order to illustrate the optimal trajectories leading into the pseudo-equilibrium, we have employed a numerical analysis. Using the parameter specification $K = 1, \rho = 0.025, p = 2, c = 1, r = 1$, Fig. 4 illustrates the optimal solution converging to the pseudo-equilibrium for both $x_0 < x_s$ and $x_0 > x_s$. The pseudo-equilibrium is the point (0.4, 1.97).

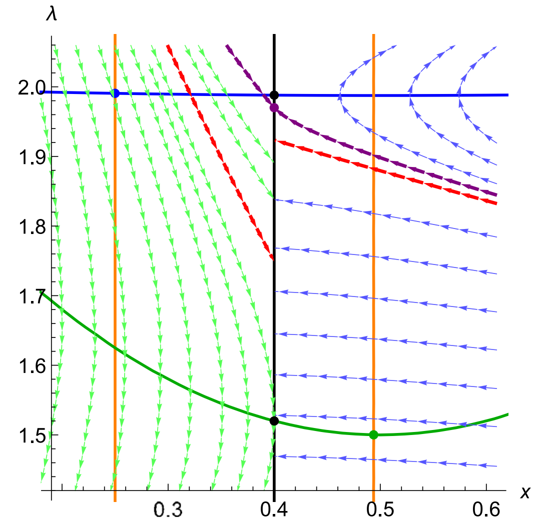

Fig. 4. Phase diagram for the case $r_+ = 0.05 < 2 = r_-$ with $x_s = 0.4$. The bounds $\lambda_{min} = 1.52$ and $\lambda_{max} = 1.988$ are indicated by black points on the switching manifold; the pseudo-equilibrium $(x_s, \lambda_\ell^\infty) = (0.4, 1.97)$ and the associated paths from the left and the right are all displayed in purple.. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Since neither of the CSS can be reached, the only candidate for an optimal path is to follow the stable manifold of either X_-^∞, if $x(t) < x_s$, or X_+^∞, if $x(t) > x_s$, entering the switching manifold in finite time. Fig. 4, which is a zoom of Fig. 2(b), reveals that for points on the switching manifold $(x_s, \lambda) \in \Sigma$, with $\lambda \in \Lambda_{SL} \equiv [\lambda_{min}, \lambda_{max}]$, where λ_{min} is the solution of $\dot{x}_- = 0$ and λ_{max} is the solution of $\dot{x}_+ = 0$, there are trajectories entering the switching manifold from both sides.

Hence, we find that, unlike the uncontrolled dynamical systems considered in Qin, Tan, Shi, Chen, and Liu (2019), Tan, et al. (2016), Tang, Liang, Xiao, and Cheke (2012), Tang et al. (2014b), Zhang and Tang (2013), among others, where sliding dynamics is a temporary phenomenon, here *optimality* requires that once we enter the sliding region, the system *remains* there. This happens because there are no other viable options for the long-run dynamics when all CSS are virtual. In order to show this more formally, we first confirm, in the next section, that the sliding region Σ_{SL} is non-empty; then, we determine the parameter values that bring about sliding dynamics, and we then characterise sliding dynamics in λ by means of Filippov's method (see, for example, Utkin, 2015).

Proposition 17. *In the harvesting model with regime-dependent growth rates (6), the following outcomes are optimal:*

(1) Once $r_- < r_+$, it follows $x_+^\infty > x_-^\infty$ and:

- *(a) If $x_+^\infty > x_s > x_-^\infty$, either X_+^∞ or X_-^∞ are reached, depending on x_0;*
- *(b) If $x_+^\infty > x_-^\infty > x_s$, only X_+^∞ is optimal;*
- *(c) If $x_s > x_+^\infty > x_-^\infty$, only X_-^∞ is optimal;*

(2) Once $r_- > r_+$, it follows $x_+^\infty < x_-^\infty$ and:

- *(a) If $x_-^\infty > x_+^\infty > x_s$, only X_+^∞ is optimal;*
- *(b) If $x_s > x_-^\infty > x_+^\infty$, only X_-^∞ is optimal;*
- *(c) If $x_-^\infty > x_s > x_+^\infty$ and the pseudo-equilibrium is weakly feasible, then convergence to the boundary stock value x_s at the (unstable) steady state of λ_ℓ, and thus to $X_\ell^\infty = (x_s, \lambda_\ell)$, is the only possible optimal outcome.*

Weak feasibility is an essential condition for this result. If the pseudo-equilibrium were strongly feasible, which according to Corollary 15 cannot be the case here, there is more than one way to reach X_ℓ^∞. In particular, one could reach X_ℓ^∞ by entering the sliding region at an arbitrary point (x_s, λ) with $\lambda \in \Lambda_{SL}$, and then proceed via the Filippov's flow $g_\ell(\lambda)$ towards the pseudo-equilibrium. We thus would have a continuum of potential trajectories leading to X_ℓ^∞, and the selection of the optimal trajectory needs to be based on an evaluation of the objective function. Moreover, in this situation we would have to deal with possibility of *chattering* dynamics in the neighbourhood of the switching manifold. These questions are left for future extensions of the current study.

6. Conclusions

In this paper we contribute to the theory of optimal harvesting of a renewable resource. Examples may include natural bio-resources (*e.g.* fish, forest), but this type of model may be also applied to problems of groundwater mining, hydropower and other issues. We extend a standard harvesting model to a piecewise-smooth (PWS) system, where the switch is supposed to occur in the growth rate of the stock, once a specified stock level is exceeded. We find that there are at least two novel types of behaviour, not previously studied in the literature, which emerge due to the hybrid nature of the control problem we study: First, we may have tipping points, similar to the well-known Skiba (or DNSS) points for smooth systems (see Wagener, 2003, for example), yet with the difference that here these points do not result from the non-linearity of the dynamics, but are due to the fact that each of the two regimes of the system dynamics has its own steady states: Even if each of the two regimes has a unique steady state with the saddle-point property, the extended *piecewise-smooth system* (PWS system) might have multiple steady states — and if at least two of them are regular, we might observe this type of pseudo-Skiba (pseudo DNSS) point. Secondly, if each of the two regimes has only a virtual steady state, implying that none of steady states can be reached from either regime, then a sliding flow at the switching manifold (threshold level) exists; this sliding flow preserves the threshold level of the stock, yet with a varying costate variable (shadow price of the stock), and thus has its own steady state, to which we refer as a *pseudo-equilibrium* of the system. We show that whenever a steady state of the PWS system cannot be reached, this type of a pseudo-equilibrium is the unique optimal outcome — even if it is unstable. Yet, the actual optimal harvesting policy depends on the initial stock.

Acknowledgements

The work of A. Bondarev was supported by the Key Program Special Fund of Xi'an Jiaotong-Liverpool University (grant no. KSF-E-63).

Appendix. Proofs

Proof of Lemma 4

(1) If $r_- < r_+$, then by (10) $x_-^\infty < x_+^\infty$, and we have:

- (i) If $x_-^\infty < x_s < x_+^\infty$, both CSS are regular and can be reached. One of them is selected, depending on the location of the initial stock x_0 relative to the location of the switching level x_s; moreover, no hybrid limit cycles may emerge (see Reddy, Schumacher, & Engwerda, 2020).
- (ii) If $x_-^\infty < x_s$ and $x_+^\infty < x_s$, only X_-^∞ is regular, and it can be reached either without a switch if $x_0 < x_s$, or with a single switch if $x_0 > x_s$.
- (iii) If $x_-^\infty > x_s$ and $x_+^\infty > x_s$, only X_+^∞ is regular, and it can be reached either without a switch or with a single switch.

(2) If $r_- > r_+$, then by (10) $x_-^\infty > x_+^\infty$, and we have:

- (i) If both X_-^∞ and X_+^∞ are located above the threshold, *i.e.* $x_-^\infty > x_+^\infty > x_s$, only X_+^∞ is regular, and it can be reached either without a switch (if $x_0 > x_s$) or with a single switch (if $x_0 < x_s$).
- (ii) If both X_-^∞ and X_+^∞ are located below the threshold, *i.e.* $x_s > x_-^\infty > x_+^\infty$, only X_-^∞ is regular, and it can be reached either without a switch (if $x_0 < x_s$) or with a single switch (if $x_0 > x_s$).
- (iii) If $x_-^\infty > x_s > x_+^\infty$, then both X_-^∞ and X_+^∞ are virtual and thus neither of them can be reached.

Proof of Corollary 8

By (14) there are only two equilibria for g_ℓ, one of which is trivial and cannot lie within the sliding region. Hence there is a unique positive steady state λ_ℓ^∞. Moreover, λ_ℓ^∞ cannot be reached from any x_0 if $\lambda_\ell^\infty \notin \Lambda_{SL}$; *i.e.* if $\lambda_\ell^\infty \notin \Lambda_{SL}$, λ_ℓ^∞ is a virtual steady state.

Proof of Lemma 10

It follows from the preceding analysis that the steady states of (8), X_-^∞ and X_+^∞, are both virtual if $x_-^\infty > x_s > x_+^\infty$. Then, if $\lambda_\ell^\infty \notin \Lambda_{SL}$, the pseudo-equilibrium cannot be reached from either the outside or from the inside of the sliding dynamics. Thus, even if a sliding region exists and the sliding dynamics is well-defined, there is no reachable steady state, and thus optimal dynamics do not exist, except from trivial ones.

Proof of Lemma 11

By definition of λ_{min} and λ_{max} as the bounds of the sliding region, we have either $\dot{x} = f_+ > 0, \dot{x} = f_- > 0$ or $\dot{x} = f_+ < 0, \dot{x} = f_- < 0$ in the neighbourhood of λ_{min} and λ_{max}, since the escaping region cannot co-exist with the sliding region for a 1D switching manifold (see, *e.g.* Di Bernardo, et al., 2008, for a detailed proof). It follows that the exit can be accomplished only via entering the crossing region. Also, we must have $r_+ < r_-$, since otherwise there is no sliding dynamics for the CS (8). Using $r_+ < r_-$, it follows from the state dynamics in (8) that $f_+ > 0 \Rightarrow f_- > 0$, while $f_- < 0 \Rightarrow f_+ < 0$. Hence, in the crossing region we have

$$\begin{aligned} & (f_+ > 0 \wedge f_- > 0) \quad \vee \quad (f_+ < 0 \wedge f_- < 0) \\ \Leftrightarrow \quad & f_+ > 0 \quad \vee \quad f_- < 0 \\ \Leftrightarrow \quad & \lambda(t) > p - \gamma(r_+) \quad \vee \quad \lambda(t) < p - \gamma(r_-). \end{aligned}$$

where $\gamma(r) := (crx(t)(K - x(t)))\,/K$.

Proof of Lemma 12

Replacing λ by zero in (15), we get an inconsistent inequality with both sides being positive (for $0 < x_s < K$). So, the trivial pseudo-equilibrium is not feasible.

Proof of Lemma 13

(i) Since $r_- > r_+ > \rho$ by assumption, it follows from the fact that κ is single-valued and monotonously increasing that the interval $[\kappa(r_+), \kappa(r_-)] \subset [0, K/2]$ exists and is unique.

(ii) $\lambda_{min} < \lambda_\ell^\infty$: Substituting λ_ℓ^∞ from (14) into (16) yields

$$\begin{aligned}\lambda_{min} < \lambda_\ell^\infty &\Leftrightarrow (K - 2x_s)\,(K\,(r_- - \rho) - 2r_- x_s) > 0 \\ &\Leftrightarrow (x_s < K/2 \;\wedge\; x_s < \kappa(r_-)) \\ &\qquad \vee\;(x_s > K/2 \;\wedge\; x_s > \kappa(r_-)) \\ &\Leftrightarrow x_s < \kappa(r_-) \;\vee\; x_s > K/2,\end{aligned}$$

where we used $\kappa(r_-) < K/2$.

(iii) $\lambda_{max} > \lambda_\ell^\infty$: Similarly, we have

$$\begin{aligned}\lambda_{max} > \lambda_\ell^\infty &\Leftrightarrow (K - 2x_s)\,(K\,(r_+ - \rho) - 2r_+ x_s) < 0 \\ &\Leftrightarrow (x_s < K/2 \;\wedge\; x_s > \kappa(r_+)) \\ &\qquad \vee\;(x_s > K/2 \;\wedge\; x_s < \kappa(r_+)) \\ &\Leftrightarrow \kappa(r_+) < x_s < K/2,\end{aligned}$$

where we used $\kappa(r_+) < K/2$, and the fact that $x_s > K/2 \wedge x_s < \kappa(r_+)$ is contradictory.

(iv) Combining (ii) and (iii), it follows that

$$\begin{aligned}&\lambda_{min} < \lambda_\ell^\infty < \lambda_{max} \\ \Leftrightarrow\quad & x_s < \kappa(r_-) \vee x_s > K/2 \;\wedge\; \kappa(r_+) < x_s < K/2 \\ \Leftrightarrow\quad & x_s < \kappa(r_-) \;\wedge\; \kappa(r_+) < x_s < K/2 \\ \Leftrightarrow\quad & \kappa(r_+) < x_s < \kappa(r_-).\end{aligned}$$

Combining (i) and (iv) completes the proof since by definition, X_ℓ^∞ is a pseudo-equilibrium for any x_s.

Proof of Lemma 14

By Lemma 13, for any $x_s \in (\kappa(r_+), \kappa(r_-)) \subset [0, K/2]$, the pseudo-equilibrium X_ℓ^∞ is feasible. From (14) we obtain

$$p \gtreqless \rho c x_s \psi \quad\Leftrightarrow\quad \frac{p}{c x_s \psi} - \rho \gtreqless 0.$$

Consequently, if Assumption 6 holds, the derivative (17) is positive, rendering λ_ℓ^∞ unstable, which combined with Lemma 13 proves that X_ℓ^∞ is weakly feasible. In contrast, if Assumption fails, the derivative (17) is negative, rendering λ_ℓ^∞ stable, and thus by Lemma 13, X_ℓ^∞ is strongly feasible.

Proof of Corollary 15

Assumption 6 implies that the pseudo-equilibrium X_ℓ^∞ has a positive λ_ℓ^∞ value, and that it is weakly feasible if, and only if, $x_s \in (\kappa(r_-), \kappa(r_+))$, due to Lemma 14. Hence, $X_\ell^\infty \in \mathcal{X}_{\mathrm{WF}}$ and $\mathcal{X}_{\mathrm{SF}} = \emptyset$.

Proof of Lemma 16

The condition (18) requires the optimal trajectory $x(t)$ to contact the switching manifold at time t_s at the pseudo-equilibrium point. Assume this is not the case. Then if $x_-^\infty > x_s > x_+^\infty$, any trajectory of either CS_+ or CS_- will diverge either to infinity or to zero for any control candidate from the maximum principle with transversality condition. This proves part 1. The value of the objective applied to a candidate trajectory must be finite and positive. Once we require a finite non-zero value, the only way to achieve this is to select the trajectory leading to λ_ℓ^∞. Condition (18) is sufficient and necessary for this: for any x_0, it would select the (unique) trajectory of the associated flow which leads into pseudo-equilibrium.

Proof of Proposition 17

Amounts to the application of Lemmas 4–16.

References

Aniţa, S., Behringer, S., & Moşneagu, T. (2019). Optimal harvesting of a spatially distributed renewable resource with endogenous pricing. *Mathematical Modelling of Natural Phenomena*, *14*(1), 1–13.

Behringer, S., & Upmann, T. (2014). Optimal harvesting of a spatial renewable resource. *Journal of Economic Dynamics and Control*, *42*, 105–120.

Belyakov, A. O., Davydov, A. A., & Veliov, V. M. (2015). Optimal cyclic exploitation of renewable resources. *Journal of Dynamical and Control Systems*, *21*(3), 475–494.

Belyakov, A. O., & Veliov, V. M. (2014). Constant versus periodic fishing: Age structured optimal control approach. *Mathematical Modelling of Natural Phenomena*, *9*(4), 20–37.

Bhattacharyya, J., Piiroinen, P. T., & Banerjee, S. (2021). Dynamics of a Filippov predator–prey system with stage-specific intermittent harvesting. *Nonlinear Dynamics*, *105*(1), 1019–1043.

Bhattacharyya, J., Roelke, D. L., Pal, S., & Banerjee, S. (2019). Sliding mode dynamics on a prey–predator system with intermittent harvesting policy. *Nonlinear Dynamics*, *98*(2), 1299–1314.

Bondarev, A., & Gromov, D. (2021). On the structure and regularity of optimal solutions in a differential game with regime switching and spillovers. In V. Veliov, J. Haunschmied, R. Kovacevic, & W. Semmler (Eds.), *Dynamic economic models with regime switches*. Berlin: Springer-Verlag.

Colombo, A., & Jeffrey, M. R. (2011). Nondeterministic chaos, and the two-fold singularity in piecewise smooth flows. *SIAM Journal on Applied Dynamical Systems*, *10*(2), 423–451.

Cruz-Rivera, E., Ramirez, C. H., & Vasilieva, O. (2019). Catch-to-stock dependence: The case of small pelagic fishery with bounded harvesting effort. *Natural Resource Modeling*, *32*(1), Article e12193.

Dasgupta, P., Mitra, T., & Sorger, G. (2019). Harvesting the commons. *Environmental and Resource Economics*, *72*(3), 613–636.

de Frutos, J., & Martin-Herran, G. (2019). Spatial effects and strategic behavior in a multiregional transboundary pollution dynamic game. *Journal of Environmental Economics and Management*, *97*(SI), 182–207.

Di Bernardo, M., Budd, C., Champneys, A., Kowalczyk, P., Nordmark, A., Olivar, G., et al. (2008). Bifurcations in nonsmooth dynamical systems. *SIAM Review*, *50*(4), 629–701.

Dubey, B., Agarwal, S., & Kumar, A. (2018). Optimal harvesting policy of a prey–predator model with Crowley–Martin-type functional response and stage structure in the predator. *Nonlinear Analysis: Modelling and Control*, *23*(4), 493–514.

Fabbri, G., Faggian, S., & Freni, G. (2020). Policy effectiveness in spatial resource wars: A two-region model. *Journal of Economic Dynamics and Control*, *111*, Article 103818.

Filippov, A. F. (1988). *Differential equations with discontinuous righthand sides*. Dordrecht, The Netherlands: Kluwer.

Grass, D., Caulkins, J. P., Feichtinger, G., Tragler, G., & Behrens, D. A. (2008). *Optimal control of nonlinear processes: with applications in drugs, corruption, and terror*. Berlin: Springer.

Grass, D., Uecker, H., & Upmann, T. (2019). Optimal fishery with coastal catch. *Natural Resource Modeling*, *32*(4), Article e12235.

Herrera, G. E., Moeller, H. V., & Neubert, M. G. (2016). High-seas fish wars generate marine reserves. *Proceedings of the National Academy of Sciences of the United States of America*, *113*(14), 3767–3772.

Jacquemard, A., Teixeira, M. A., & Tonon, D. J. (2013). Stability conditions in piecewise smooth dynamical systems at a two-fold singularity. *Journal of Dynamical and Control Systems*, *19*(1), 47–67.

Mitra, T., & Sorger, G. (2014). Extinction in common property resource models: An analytically tractable example. *Economic Theory*, *57*(1), 41–57.

Moberg, E. A., Pinsky, M. L., & Fenichel, E. P. (2019). Capital investment for optimal exploitation of renewable resource stocks in the age of global change. *Ecological Economics*, *165*, Article 106335.

Qin, W., Tan, X., Shi, X., Chen, J., & Liu, X. (2019). Dynamics and bifurcation analysis of a filippov predator–prey ecosystem in a seasonally fluctuating environment. *International Journal of Bifurcation and Chaos*, *29*(2), Article 1950020.

Reddy, P. V., Schumacher, J. M., & Engwerda, J. (2020). Analysis of optimal control problems for hybrid systems with one state variable. *SIAM Journal on Control and Optimisation*, *58*(6), 3262–3292.

Shaikh, M. S., & Caines, P. E. (2007). On the hybrid optimal control problem: Theory and algorithms. *IEEE Transactions on Automatic Control*, *52*(9), 1587–1603.

Tan, X., Qin, W., Liu, X., Yang, J., & Jiang, S. (2016). Sliding bifurcation analysis and global dynamics for a filippov predator–prey system. *Journal of Nonlinear Sciences and Applications*, *9*(6), 3948–3961.

Tang, S., Liang, J., Xiao, Y., & Cheke, R. A. (2012). Sliding bifurcations of Filippov two stage pest control models with economic thresholds. *SIAM Journal of Applied Mathematics*, *72*(4), 1061–1080.
Tang, G., Qin, W., & Tang, S. (2014a). Complex dynamics and switching transients in periodically forced Filippov prey–predator system. *Chaos, Solitons & Fractals*, *61*, 13–23.
Tang, S., Tang, G., & Qin, W. (2014b). Codimension-1 sliding bifurcations of a Filippov pest growth model with threshold policy. *International Journal of Bifurcation and Chaos*, *24*(10), Article 1450122.
Upmann, T., & Behringer, S. (2020). Harvesting a remote renewable resource. *Theoretical Ecology*, *13*, 459–480.
Upmann, T., Uecker, H., Hammann, L., & Blasius, B. (2021). Optimal stock–enhancement of a spatially distributed renewable resource. *Journal of Economic Dynamics and Control*, *123*, Article 104060.
Utkin, V. I. (1992). *Sliding modes in control and optimization*. Berlin: Springer-Verlag.
Utkin, V. I. (2015). Brief comments for the continuation method by AF Filippov for solution continuation on a discontinuity set. *Automation and Remote Control*, *76*(5), 863–871.
Wagener, F. O. O. (2003). Skiba points and heteroclinic bifurcations, with applications to the shallow lake system. *Journal of Economic Dynamics and Control*, *27*(9), 1533–1561.
Zhang, X., & Tang, S. (2013). Filippov ratio-dependent prey–predator model with threshold policy control. *Abstract and Applied Analysis*, *2013*, Article 280945.

Anton Bondarev is an Associate professor in Economics at IBSS. He acquired his Ph.D. from Bielefeld University in 2011 concentrating on dynamics of innovations and optimal management of R&D.

Prior to joining XJTLU he served as a Postdoc in different international projects, such as Climate Policy of Nations in Germany, Competence Center for Research in Energy, Society and Transition in Switzerland and others.

Anton's research is focused on endogenous growth, environment, promotion of innovations and optimal governance of large-scale technological transitions, be it renewable energy or modernisation of the economy.

He actively publishes in such journals as European Economic Review (ABDC: A*), Journal of Economic Dynamics and Control (ABDC: A*), China Economic Review (ABDC: A), Energy Policy (ABDC: A), Macroeconomic Dynamics (ABDC: A), Journal of Evolutionary Economics (ABDC: A) and others.

Thorsten Upmann is a senior researcher at Carl von Ossietzky University Oldenburg and is also affiliated with The Helmholtz Institute for Functional Marine Biodiversity at the University of Oldenburg (HIFMB) and with Bielefeld University in Bielefeld, all Germany. He is the group leader of the Value of Information group consisting of researchers and Ph.D. students from these three institutions.

Thorsten holds a Ph.D. and a Habilitation in Economics. His research focuses on the management of renewable natural resources, the management of ecological and environmental regime switches, the economics of fisheries, sequential decision making in environmental management and on environmental taxation.

Thorsten is an Associate Editor of the Economics Bulletin, and a member of the Editorial Board of SN Business & Economics. Moreover, he is a member of the international CESifo Research Network, Munich.

Thorsten has published 38 articles in refereed international journals, among which are Economic Inquiry, Economic Theory, Economics Letters, the European Economic Review, the Journal of Economic Dynamics and Control, Mathematical Social Sciences, Natural Resource Modeling, the Scandinavian Journal of Economics, Social Choice and Welfare, Theoretical Ecology. Also, Thorsten regularly serves as a referee for more than 40 journals.

ORCID: https://orcid.org/0000-0001-7804-7982
Publons/Web of Science: https://publons.com/researcher/AAA-9925-2019/