About domain localization in ensemble based Kalman filter algorithms

T.Janjić, L.Nerger,J.Schröter

Alfred Wegener Institute

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Outline

- Basic properties of covariance localization or direct forecast error localization (used in Houtekamer and Mitchell (1998, 2001))
- Basic aspects of domain localization (used in Haugen and Evensen 2002; Brusdal et al. 2003; Evensen 2003; Brankart et al. 2003; Ott et al. 2004; Nerger et al. 2006; Hunt et al. 2007; Miyoshi and Yamane 2007)
- Why is domain localization used?
- Ways of including full rank, positive definite and isotropic matrix in domain localized algorithms
- Lorenz40 model comparisons
- Conclusion

Direct forecast error localization or covariance localization

Covariance localization: The ensemble derived forecast error covariance matrix is Schur multiplied with a stationary a priori chosen covariance matrix that is compactly supported.

Let $\mathbf{C}$ be a matrix of rank M that is used for the Schur product. Let $\mathbf{v}_j$ represent eigenvectors of matrix $\mathbf{C}$.

$$\mathbf{P}_k^f = \frac{1}{r}\sum_{i=1}^{r+1}[\mathbf{x}^{f,i}(t_k) - \mathbf{x}_k^f][\mathbf{x}^{f,i}(t_k) - \mathbf{x}_k^f]^T.$$

$\mathbf{P}_k^f$ is the ensemble derived forecast error covariance;
$\mathbf{x}^{f,i}(t_k)$ are ensemble members $i = 1, \ldots, r+1$ of size n at time t_k;
$\mathbf{x}_k^f$ is the average over ensemble.

Covariance localization

Basic properties:

- The localized error covariance $\mathbf{P}_k^f \circ \mathbf{C}$ can be represented as

$$\sum_{i,j=1}^{r+1,M} \mathbf{u}_{i,j}\mathbf{u}_{i,j}^T \quad \text{with } \mathbf{u}_{i,j} = \frac{1}{\sqrt{r}}[\mathbf{x}^{f,i}(t_k) - \mathbf{x}_k^f] \circ \mathbf{v}_j$$

  This representation implies that instead of using ensemble members $\mathbf{x}^{f,i}$ for the calculation of the analysis error covariance, we can use the ensemble $\mathbf{u}_{i,j}$, and the same formulas as in original algorithms apply.
- $\mathbf{C}$ full rank, positive definite, isotropic matrix, compactly supported. Usually 5th order polynomial correlation function (Gaspari and Cohn 1999).
- $min(diag(\mathbf{P}_k^f))\lambda_{min}(\mathbf{C}) \leq \lambda_{min}(\mathbf{P}_k^f \circ \mathbf{C}) \leq \lambda_{max}(\mathbf{P}_k^f \circ \mathbf{C}) \leq max(diag(\mathbf{P}_k^f))\lambda_{max}(\mathbf{C})$

Domain localization

Domain localization: Disjoint domains in the physical space are considered as domains on which the analysis is performed. Therefore, for each subdomain an analysis step is performed independently using observations not necessarily belonging only to that subdomain. Results of the local analysis steps are pasted together and then the global forecast step is performed.

Basic properties:

- The localized error covariance is calculated using

$$\mathbf{P}_k^{f,loc} = \sum_{i,j=1}^{r+1,L} \mathbf{u}_{i,j}\mathbf{u}_{i,j}^T \tag{1}$$

where $\mathbf{u}_{i,j} = \frac{1}{\sqrt{r}}[\mathbf{x}^{f,i}(t_k) - \mathbf{x}_k^f] \circ \mathbf{1}_{Dj}$ with $j = 1, \ldots, L$ and L is the number of subdomains. Here $\mathbf{1}_{Dj}$ is a vector whose elements are 1 if the corresponding point belongs to the domain Dj.

Domain localization

- $\mathbf{C}$ positive semidefinite, has block structure and is the sum of rank one matrices $\mathbf{1}_{Dj}\mathbf{1}_{Dj}^T$. The rank of matrix $\mathbf{C}$ corresponds to the number of subdomains.
- In case that $\text{rank}(\mathbf{C})\text{rank}(\mathbf{P}_k^f) < n$, the matrix $\mathbf{C} \circ \mathbf{P}_k^f$ is singular.

Why is domain localization used?

- As for OI, one of the major advantages of using domain localization is computational. The updates on the smaller domains can be done independently, and therefore in parallel.
- In certain algorithms this is more natural way of localizing. Examples of such methods are the ensemble transform Kalman filter ETKF and the singular evolutive interpolated Kalman filter SEIK.

Why is domain localization used?

- In these algorithms, the forecast error covariance matrix is never explicitly calculated. Therefore, direct forecast localization as in HM98,HM01 is not immediately possible.
- In these methods an ensemble resampling in SEIK or transformation is used that ensures that the ensemble statistics represent exactly the analysis state and error covariance matrix.
- Ways of including full rank, positive definite and isotropic matrix in domain localized algorithms were developed. Two methods will be presented Method SD+Loc and Method SD+ObsLoc introduced by Hunt et al. 2007.

Method SD+Loc

Let $\mathbf{1}_{Dmj}$ be a vector that has a value of 1 if the observation belongs to the domain Dm otherwise has a value of 0, and let $Dj \subseteq Dmj$.

$$\frac{1}{r}\sum_{i=1}^{r+1}\sum_{j=1}^{L}[\mathbf{H_k}\mathbf{x}^{f,i}(t_k)\circ\mathbf{1}_{Dmj} - \mathbf{H_k}\mathbf{x}_k^f\circ\mathbf{1}_{Dmj}][\mathbf{x}^{f,i}(t_k)\circ\mathbf{1}_{Dj} - \mathbf{x}_k^f\circ\mathbf{1}_{Dj}]^T$$

$$= \sum_{j=1}^{L}(\mathbf{1}_{Dmj}\mathbf{1}_{Dj}^T)\circ\mathbf{H_k}\mathbf{P}_k^f$$

where matrix $\sum_{j=1}^{L}\mathbf{1}_{Dmj}\mathbf{1}_{Dj}^T$ has entries of zeros and ones since the domains Dj are disjoint.

Method (SD+Loc): An modification to this algorithm is to use for each subdomain $(\mathbf{1}_{Dmj}\mathbf{1}_{Dj}^T)\circ\mathbf{H_k}\mathbf{P}_k^f\circ\mathbf{H_k}\mathbf{C}$ and $\mathbf{1}_{Dmj}\mathbf{1}_{Dmj}^T\circ\mathbf{H_k}\mathbf{P}_k^f\mathbf{H_k}^T\circ\mathbf{H_k}\mathbf{C}\mathbf{H_k}^T$.

Observational error localization: Method (SD+ObLoc)

The observation localization method modifies the observational error covariance matrix **R**.

Let us consider a single observation example, in observation error localization method, the observation error σ^2_{obs} is modified to $\sigma^2_{obs}/weight_d$ where $weight_d$ can be calculated using any of the correlation functions.

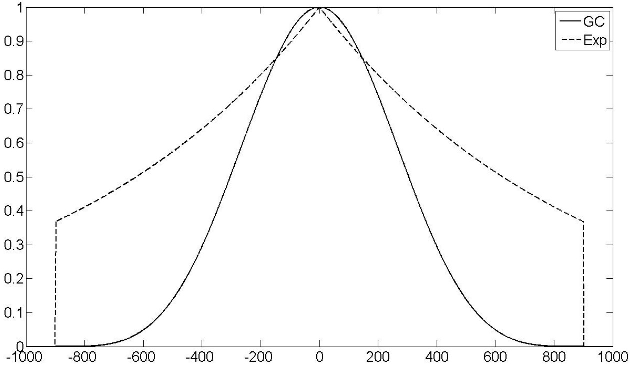
Accordingly, the analysis increment is multiplied by $weight_d pf/(weight_d + \sigma^2_{obs})$, where $weight_d$ depends on the distance between observation and analysis point.

Note, for direct forecast error localization this factor is $weight_d pf/(1 + \sigma^2_{obs})$.

Model Lorenz40

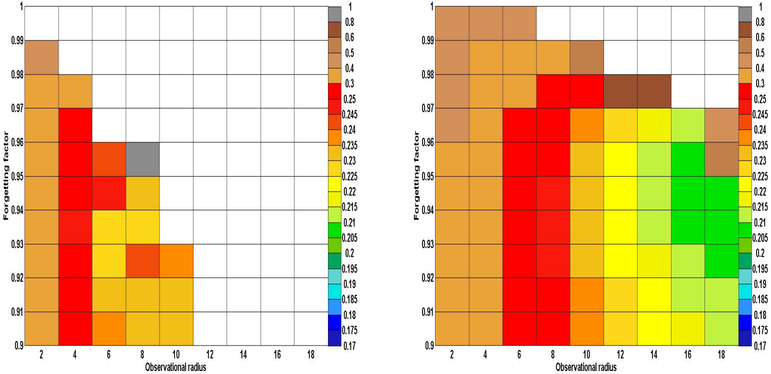
- Lorenz40 model is governed by 40 coupled ordinary differential equations in domain with cyclic boundary conditions.
- The state vector dimension is 40.
- The experimental setup follows Whitaker and Hamill (2002).
- The observations are given as a vector of values contaminated by uncorrelated normally distributed random noise with standard deviation of 1.
- The observations are assimilated at every time step.
- After a spin-up period of 1000 time steps, assimilation is performed for another 50 000 time steps.
- A 10-member ensemble is used.
- SEIK filter with localization is used.
- In this setting the SEIK filter without localization diverges for all forgetting factors shown in the experiments below.

5th order polynomial weighting



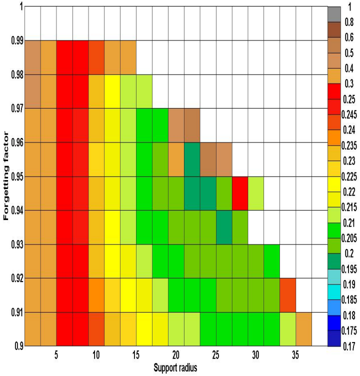
Both methods (SD+ObLoc) and (SD+Loc) use 5th order polynomial for weighting. Method (SD+) uses uniform weighting.

L40 results



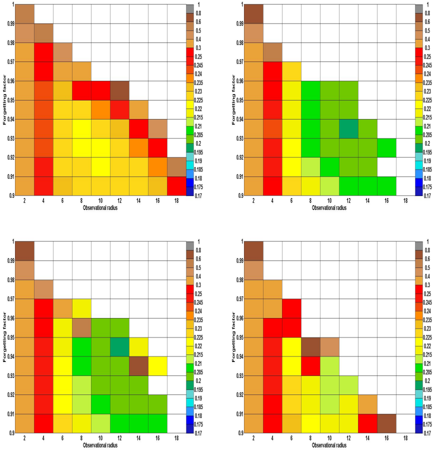
Ensemble-mean error as a function of observational radius and the forgetting factor. Results are for method (SD+) and (SD+ObLoc).

L40 results



Covariance localisation as in Whitaker and Hamill (2002) (left) and results for method (SD+ObLoc) plus support radius going beyond observational radius.

Metod SD+Loc



Method SD+Loc support radius 9 (upper left), support radius 21 (upper right), support radius 23 (lower left) and support radius 39 (lower right).

Conclusion

- The domain localization technique has been investigated here and compared to direct forecast error localization on L40 model.
- It was shown that domain localization is equivalent to direct forecast error localization with a Schur product matrix that has a block structure and is not isotropic.
- The rank of the matrix corresponding to the domain localization depends on the number of subdomains that are used in the assimilation. This matrix is positive semidefinite.
- An algorithm is presented that for each subdomain of ensemble localization uses observations from a domain larger than the ensemble subdomain and a Schur product with an isotropic matrix on each subdomain.

Conclusion

- Results obtained from a simple example show that the errors obtained with this method are comparable to the direct forecast localization technique.
- In addition, these results were compared to a method that for each subdomain of ensemble localization uses observations from a domain larger than the ensemble subdomain and applies localization of observations.